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ON THE REPRESENTATION OF CONTINUOUS FUNCTIONS OF SEVERAL VARIABLES BY SUPERPOSITIONS OF CONTINUOUS FUNCTIONS OF A SMALLER NUMBER OF VARIABLES

A. N. KOLMOGOROV

From Theorem 3, stated below, there follows a rather unexpected consequence: *every continuous function of an arbitrarily large number of variables can be represented in the form of a finite superposition of continuous functions of not more than three variables*. For an arbitrary function of four variables such a representation has the form

$$f(x_1, x_2, x_3, x_4) = \sum_{r=1}^{4} h^r[x_4, g_1^r(x_1, x_2, x_3), g_2^r(x_1, x_2, x_3)].$$

The question of the possibility of representing an arbitrary continuous function of three variables in the form of a superposition of continuous functions of two variables remains open. The proof of the possibility of such a representation would yield a complete solution of Hilbert's 13th problem [1] in the sense of establishing the hypothesis stated by Hilbert. Theorem 2 shows only that representation of an arbitrary continuous function of three variables in the form of a superposition of continuous functions of two variables is in fact possible if as auxiliary variables one admits variables running through a one-dimensional configuration somewhat more complicated than an interval of the real line, and in fact, the universal tree (a tree is a locally connected continuum not containing a homeomorphic image of a circle; as Menger [2] showed, there is a universal tree containing homeomorphic images of all trees).

In the sequel, k, m, n, r will denote natural numbers; $a, b, c, C, d, M, R, x, y, u, v, f, F, g, h, \epsilon, \delta, \rho$ will denote real numbers; ξ, ϕ, ψ will denote elements of trees; E^n will denote the n-dimensional unit cube, $0 \le x_i \le 1$; $i = 1, \cdots, n$.

Theorem 1.

a) *For arbitrary $n \ge 2$, there are continuous functions*

$$\phi^1, \cdots, \phi^{n+1}$$

defined on E^n with values in the universal tree Ξ such that every continuous real-valued function f defined on E^n has the form

$$f(x_1, \cdots, x_n) = \sum_{r=1}^{n+1} h_f^r[\phi^r(x_1, \cdots, x_n)],$$

where the real functions $h_f^r(\xi)$ are defined and continuous on Ξ.

b) *Here the functions h_f^r can be chosen so that they will depend continuously on f in the sense of the topology of uniform convergence in the spaces of continuous functions on E^n and Ξ.*

Theorem 2 follows almost immediately from Theorem 1.

Theorem 2. *For arbitrary* $n \geq 3$, *there exist functions*

$$\phi^1, \cdots, \phi^n$$

defined and continuous on E^n *with values in* Ξ, *such that every continuous function* f *defined on* E^n *is representable in the form*

$$f(x_1, \cdots, x_n) = \sum_{r=1}^{n} h^r[x_n, \phi^r(x_1, \cdots, x_{n-1})],$$

where the real functions $h^r(x, \xi)$ *are defined and continuous on the product* $E^1 \times \Xi$.

The universal tree Ξ (see [2]) can be realized in the form of a continuum lying in the unit square E^2. Writing g_1^r and g_2^r for the coordinates of the point ϕ^r, we obtain this assertion as an immediate consequence of Theorem 2:

Theorem 3. *For arbitrary* $n \geq 3$, *there exist continuous real functions*

$$g_1^1, \cdots, g_1^n;\ g_2^1, \cdots, g_2^n,$$

defined on E^n *such that every continuous function* f *given on* E^n *is representable in the form*

$$f(x_1, \cdots, x_n) = \sum_{r=1}^{n} h^r[x_n, g_1^r(x_1, \cdots, x_{n-1}), g_2^r(x_1, \cdots, x_{n-1})],$$

where the functions h^r *are defined and continuous on* E^3.

For $n = 3$, Theorem 3 is trivial: it has real interest only for $n \geq 4$.

It remains to sketch the method of proof of Theorem 1. The following lemma is the starting point for this.

Basic Lemma. *For arbitrary* $n \geq 2$, *one can define on* E^n *a system of functions*

$$u_{km}^r(x_1, \cdots, x_n)$$

with indices r, k, m *running through values in the intervals*

$$1 \leq r \leq n+1,\ 1 \leq k < \infty,\ 1 \leq m \leq m_k,$$

that possesses the following properties:

1) $u_{km}^r \geq 0$;

2) $u_{km}^r \neq 0$ *only on a set* G_{km}^r *of diameter* d_k, *where* $d_k \to 0$ *as* $k \to \infty$;

3) *two sets* G_{km}^r *and* $G_{km'}^r$, *with common indices* r *and* k *and* $m' \neq m$ *do not intersect;*

4) *for arbitrary* k *and at every point* $P \in E^n$,

$$c \leq \sum_{r=1}^{r=n+1} \sum_{m=1}^{m=m_k} u_{km}^r \leq C,$$

where c *and* C *do not depend upon* k;

5) *the function* u_{km}^r *is constant on each* $G_{k'm'}^r$ *with the same index* r *for* $k' > k$ *and arbitrary* m'.

The construction of the system of functions u_{km}^r cannot be set down within the limits of this note. In the sequel, we shall suppose that this system of functions is given.

Lemma 1.

a) *An arbitrary continuous function* f *defined on* E^n *can be represented in the form*

$$f(P) = \sum_{k=1}^{\infty} \sum_{r=1}^{n+1} \sum_{m=1}^{m_k} a_{km}^r(f)\, u_{km}^r(P), \tag{1}$$

where the coefficients $a_{km}^r(f)$ *do not depend upon* P.

b) *The coefficients* $a_{km}^r(f)$ *can be chosen in the form of continuous functions of* f *and indeed so that*

$$|a_{km}^r(f)| \leq a(\mathfrak{F}),\quad \sum_{k=1}^{\infty} a_k(\mathfrak{F}) < \infty$$

on every family $\mathfrak{F}$ of uniformly bounded and equicontinuous functions f.

The proof of Lemma 1 is based on properties 1), 2), and 4) of the system u_{km}^r and begins with estimates of the remainder term R in the representation

$$f(P) = \sum_{r=1}^{n+1} \sum_{m=1}^{m_k} b_m^r u_{mk}^r(P) + R,$$

where

$$b_m^r = \frac{1}{C} f(P_{km}^r),$$

and P_{km}^r are arbitrary points belonging respectively to the sets G_{km}^r. It is easy to show that for an appropriate choice of the coefficients b_m^r,

$$|R| \leq (|1 - \frac{c}{C}| + \delta_k) M,$$

where

$$M = \sup_{P \in E^n} |f(P)|,\quad \delta_k = \sup_{\rho(P,P') \leq d_k} |f(P) - f(P')|.$$

A complete proof of Lemma 1 cannot be given here.

We now write the decomposition (1) in the form

$$f(P) = \sum_{r=1}^{n+1} f^r(P), \tag{2}$$

$$f^r(P) = \sum_{k=1}^{\infty} \sum_{m=1}^{m_k} a_{km}^r(f)\, u_{km}^r(P).$$

The following property of the functions f^r can be easily inferred from properties 2), 3), and 5) of the system u_{km}^r.

Lemma 2. *The function* $f^r(P)$ *is constant on every component of an arbitrary level set of the function*

$$F^r(P) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{m_k} u^r_{km}(P).$$

We now remark, as was shown by A. S. Kronrod [3], that the components of level sets of an arbitrary continuous function form a tree in a certain natural topology. The tree of components of the level sets of the function F^r will be denoted by Ξ^r, and these trees $\Xi^1, \cdots, \Xi^{n+1}$ are then mapped by homeomorphisms

$$\psi_r(\Xi^r) = \Xi^r_0 \subseteq \Xi$$

onto pairwise disjoint subsets of the universal tree Ξ. We set

$$\phi^r(P) = \psi_r(\xi^r), \text{ if } P \in \xi^r \in \Xi^{r0},$$

and define continuous functions $h^r(\xi)$ on Ξ such that for $\xi \in \Xi^r_0$,

$$h^r(\xi) = y, \text{ if } f^r(P) = y \text{ for } P \in \psi_r^{-1}(\xi).$$

It is easy to verify that

$$f^r(P) = h^r[\phi^r(P)]. \tag{3}$$

Formulas (2) and (3) lead us to a proof of assertion a) of Theorem 1. Assertion b) of Theorem 1 is proved on the basis of assertion b) of Lemma 1.

In conclusion we list without proof the following assertion.

Theorem 4. *For arbitrary $n \geq 2$ and $\epsilon > 0$, for every function f defined and continuous on E^n, there exist polynomials*

$$b(u_1, \cdots, u_{n-1}),\ a_r(x),\ c_r(x);\ \ r = 1, \cdots, n+1,$$

such that at all points $P \in E^n$

$$|f(P) - \tilde{f}(P)| < \epsilon,$$

where

$$\tilde{f}(x_1, \cdots, x_n) = \sum_{r=1,2} a_r(x_n)\, b[c_r(x_n) + x_1, \cdots, c_r(x_n) + x_{n-1}]. \tag{4}$$

In the case $n = 3$, upon setting

$$d(u, v) = u + v,\ \ g_r(x, y) = a_r(x)y,\ \ h_r(x, x') = c_r(x) + x',$$

we obtain from (4)

$$\tilde{f}(x_1, x_2, x_3) =$$
$$d(g_1\{x_3, b[h_1(x_3, x_1), h_1(x_3, x_2)]\}, g_2\{x_3, b[h_2(x_3, x_1), h_2(x_3, x_2)]\}). \tag{5}$$

In view of Theorem 4, *every continuous function of three variables can be approximated arbitrarily by an expression of the form* (5), *where d, g_r, b, and h_r are polynomials of two variables.* This remark also illuminates from a rather new side the circle of problems relating to Hilbert's 13th problem.

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