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**ON THE REPRESENTATION OF CONTINUOUS FUNCTIONS  
OF SEVERAL VARIABLES BY SUPERPOSITIONS OF CONTINUOUS FUNCTIONS  
OF A SMALLER NUMBER OF VARIABLES**

A. N. KOLMOGOROV

From Theorem 3, stated below, there follows a rather unexpected consequence: every continuous function of an arbitrarily large number of variables can be represented in the form of a finite superposition of continuous functions of not more than three variables. For an arbitrary function of four variables such a representation has the form

$$f(x_1, x_2, x_3, x_4) = \sum_{r=1}^4 h^r[x_4, g_1^r(x_1, x_2, x_3), g_2^r(x_1, x_2, x_3)].$$

The question of the possibility of representing an arbitrary continuous function of three variables in the form of a superposition of continuous functions of two variables remains open. The proof of the possibility of such a representation would yield a complete solution of Hilbert's 13th problem [1] in the sense of establishing the hypothesis stated by Hilbert. Theorem 2 shows only that representation of an arbitrary continuous function of three variables in the form of a superposition of continuous functions of two variables is in fact possible if as auxiliary variables one admits variables running through a one-dimensional configuration somewhat more complicated than an interval of the real line, and in fact, the universal tree (a tree is a locally connected continuum not containing a homeomorphic image of a circle; as Menger [2] showed, there is a universal tree containing homeomorphic images of all trees).

In the sequel,  $k, m, n, r$  will denote natural numbers;  $a, b, c, C, d, M, R, x, y, u, v, f, F, g, h, \epsilon, \delta, \rho$  will denote real numbers;  $\xi, \phi, \psi$  will denote elements of trees;  $E^n$  will denote the  $n$ -dimensional unit cube,  $0 \leq x_i \leq 1; i = 1, \dots, n$ .

**Theorem 1.**

a) For arbitrary  $n \geq 2$ , there are continuous functions

$$\phi^1, \dots, \phi^{n+1}$$

defined on  $E^n$  with values in the universal tree  $\Xi$  such that every continuous real-valued function  $f$  defined on  $E^n$  has the form

$$f(x_1, \dots, x_n) = \sum_{r=1}^{n+1} h_f^r[\phi^r(x_1, \dots, x_n)],$$

where the real functions  $h_f^r(\xi)$  are defined and continuous on  $\Xi$ .

b) Here the functions  $h_f^r$  can be chosen so that they will depend continuously on  $f$  in the sense of the topology of uniform convergence in the spaces of continuous functions on  $E^n$  and  $\Xi$ .

Theorem 2 follows almost immediately from Theorem 1.

**Theorem 2.** For arbitrary  $n \geq 3$ , there exist functions

$$\phi^1, \dots, \phi^n$$

defined and continuous on  $E^n$  with values in  $\Xi$ , such that every continuous function  $f$  defined on  $E^n$  is representable in the form

$$f(x_1, \dots, x_n) = \sum_{r=1}^n h^r[x_n, \phi^r(x_1, \dots, x_{n-1})],$$

where the real functions  $h^r(x, \xi)$  are defined and continuous on the product  $E^1 \times \Xi$ .

The universal tree  $\Xi$  (see [2]) can be realized in the form of a continuum lying in the unit square  $E^2$ . Writing  $g_1^r$  and  $g_2^r$  for the coordinates of the point  $\phi^r$ , we obtain this assertion as an immediate consequence of Theorem 2:

**Theorem 3.** For arbitrary  $n \geq 3$ , there exist continuous real functions

$$g_1^1, \dots, g_1^n; g_2^1, \dots, g_2^n,$$

defined on  $E^n$  such that every continuous function  $f$  given on  $E^n$  is representable in the form

$$f(x_1, \dots, x_n) = \sum_{r=1}^n h^r[x_n, g_1^r(x_1, \dots, x_{n-1}), g_2^r(x_1, \dots, x_{n-1})],$$

where the functions  $h^r$  are defined and continuous on  $E^3$ .

For  $n = 3$ , Theorem 3 is trivial: it has real interest only for  $n \geq 4$ .

It remains to sketch the method of proof of Theorem 1. The following lemma is the starting point for this.

**Basic Lemma.** For arbitrary  $n \geq 2$ , one can define on  $E^n$  a system of functions

$$u_{km}^r(x_1, \dots, x_n)$$

with indices  $r, k, m$  running through values in the intervals

$$1 \leq r \leq n+1, 1 \leq k < \infty, 1 \leq m \leq m_k,$$

that possesses the following properties:

- 1)  $u_{km}^r \geq 0$ ;
- 2)  $u_{km}^r \neq 0$  only on a set  $G_{km}^r$  of diameter  $d_k$ , where  $d_k \rightarrow 0$  as  $k \rightarrow \infty$ ;
- 3) two sets  $G_{km}^r$  and  $G_{k'm'}^r$ , with common indices  $r$  and  $k$  and  $m' \neq m$  do not intersect;

4) for arbitrary  $k$  and at every point  $P \in E^n$ ,

$$c \leq \sum_{r=1}^{r=n+1} \sum_{m=1}^{m=m_k} u_{km}^r \leq C,$$

where  $c$  and  $C$  do not depend upon  $k$ ;

5) the function  $u_{km}^r$  is constant on each  $G_{k'm'}^r$  with the same index  $r$  for

$k' > k$  and arbitrary  $m'$ .

The construction of the system of functions  $u_{km}^r$  cannot be set down within the limits of this note. In the sequel, we shall suppose that this system of functions is given.

**Lemma 1.**

a) An arbitrary continuous function  $f$  defined on  $E^n$  can be represented in the form

$$f(P) = \sum_{k=1}^{\infty} \sum_{r=1}^{n+1} \sum_{m=1}^{m_k} a_{km}^r(f) u_{km}^r(P), \quad (1)$$

where the coefficients  $a_{km}^r(f)$  do not depend upon  $P$ .

b) The coefficients  $a_{km}^r(f)$  can be chosen in the form of continuous functions of  $f$  and indeed so that

$$|a_{km}^r(f)| \leq a(\mathfrak{F}), \quad \sum_{k=1}^{\infty} a_k(\mathfrak{F}) < \infty$$

on every family  $\mathfrak{F}$  of uniformly bounded and equicontinuous functions  $f$ .

The proof of Lemma 1 is based on properties 1), 2), and 4) of the system  $u_{km}^r$  and begins with estimates of the remainder term  $R$  in the representation

$$f(P) = \sum_{r=1}^{n+1} \sum_{m=1}^{m_k} b_m^r u_{mk}^r(P) + R,$$

where

$$b_m^r = \frac{1}{C} f(P_{km}^r),$$

and  $P_{km}^r$  are arbitrary points belonging respectively to the sets  $G_{km}^r$ . It is easy to show that for an appropriate choice of the coefficients  $b_m^r$ ,

$$|R| \leq (|1 - \frac{c}{C}| + \delta_k) M,$$

where

$$M = \sup_{P \in E^n} |f(P)|, \quad \delta_k = \sup_{\rho(P, P') \leq d_k} |f(P) - f(P')|.$$

A complete proof of Lemma 1 cannot be given here.

We now write the decomposition (1) in the form

$$f(P) = \sum_{r=1}^{n+1} f^r(P), \quad (2)$$

$$f^r(P) = \sum_{k=1}^{\infty} \sum_{m=1}^{m_k} a_{km}^r(f) u_{km}^r(P).$$

The following property of the functions  $f^r$  can be easily inferred from properties 2), 3), and 5) of the system  $u_{km}^r$ .

**Lemma 2.** The function  $f^r(P)$  is constant on every component of an arbitrary level set of the function

$$F^r(P) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{mk} u_{km}^r(P).$$

We now remark, as was shown by A. S. Kronrod [3], that the components of level sets of an arbitrary continuous function form a tree in a certain natural topology. The tree of components of the level sets of the function  $F^r$  will be denoted by  $\Xi^r$ , and these trees  $\Xi^1, \dots, \Xi^{n+1}$  are then mapped by homeomorphisms

$$\psi_r(\Xi^r) = \Xi_0^r \subseteq \Xi$$

onto pairwise disjoint subsets of the universal tree  $\Xi$ . We set

$$\phi^r(P) = \psi_r(\xi^r), \text{ if } P \in \xi^r \in \Xi_0^r,$$

and define continuous functions  $h^r(\xi)$  on  $\Xi$  such that for  $\xi \in \Xi_0^r$ ,

$$h^r(\xi) = y, \text{ if } f^r(P) = y \text{ for } P \in \psi_r^{-1}(\xi).$$

It is easy to verify that

$$f^r(P) = h^r[\phi^r(P)]. \quad (3)$$

Formulas (2) and (3) lead us to a proof of assertion a) of Theorem 1. Assertion b) of Theorem 1 is proved on the basis of assertion b) of Lemma 1.

In conclusion we list without proof the following assertion.

**Theorem 4.** For arbitrary  $n \geq 2$  and  $\epsilon > 0$ , for every function  $f$  defined and continuous on  $E^n$ , there exist polynomials

$$b(u_1, \dots, u_{n-1}), a_r(x), c_r(x); r = 1, \dots, n+1,$$

such that at all points  $P \in E^n$

$$|f(P) - \tilde{f}(P)| < \epsilon,$$

where

$$\tilde{f}(x_1, \dots, x_n) = \sum_{r=1,2} a_r(x_n) b[c_r(x_n) + x_1, \dots, c_r(x_n) + x_{n-1}]. \quad (4)$$

In the case  $n = 3$ , upon setting

$$d(u, v) = u + v, g_r(x, y) = a_r(x)y, h_r(x, x') = c_r(x) + x',$$

we obtain from (4)

$$\tilde{f}(x_1, x_2, x_3) =$$

$$d(g_1\{x_3, b[h_1(x_3, x_1), h_1(x_3, x_2)]\}, g_2\{x_3, b[h_2(x_3, x_1), h_2(x_3, x_2)]\}). \quad (5)$$

In view of Theorem 4, every continuous function of three variables can be approximated arbitrarily by an expression of the form (5), where  $d, g_r, b$ , and  $h_r$  are polynomials of two variables. This remark also illuminates from a rather new side the circle of problems relating to Hilbert's 13th problem.

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