Quadratic Inverse Eigenvalue Problems, Active Vibration Control and Model Updating

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Abstract

This paper presents a brief review of recent developments on quadratic inverse eigenvalue problem with applications to active vibration control and finite element model updating.

1. Introduction

The Quadratic Inverse Eigenvalue Problem (QIEP) is to construct three matrices, M, C, and, K, from a given set of numbers and vectors such that these numbers belong to the spectrum of the quadratic matrix pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ with the given vectors as eigenvectors. For convenience, we denote this pencil by (M, C, K). The QIEP is an important practical problem that arises in a wide range of applications, including mechanical vibrations, acoustic studies, signal processing, fluid dynamics, and control theory. In a realistic situation, only a few eigenvalues and the associated eigenvectors of the pencil $P(\lambda)$ can be computed or measured. This remark

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is specially applicable to large and sparse models, which very often arise in practical applications. It is thus particularly important and challenging to solve QIEP from the knowledge of the partial spectrum and associated eigenvectors. Indeed, two important variations of such partially prescribed quadratic eigenvalue problems arise in **active vibration control (AVC)** and **finite element model updating (FEMU)** in mechanical vibration.

The **active vibration control** refers to controlling vibrations in structures, caused by a few undesirable frequencies, by using an active control force. The most important part of active control implementation is to effectively compute the feedback control matrices. Computation of feedback matrices for AVC is *the quadratic partial eigenvalue assignment problem* (QPEVAP). The problem is to compute two feedback matrices F and G, given a small set of eigenvalues and eigenvectors and a control matrix B, such that these eigenvalues are contained in the spectrum of the controlled pencil $\lambda^2 M + \lambda(C - BF^T) + (K - BG^T)$ with the given vectors as associated eigenvectors, while the remaining large number of eigenvalues and eigenvectors remain unchanged. The last requirement is known as the *no spill-over* phenomenon.

Finite element model updating concerns with updating of the finite element model, using a few measured eigenvalues and eigenvectors, such that the modified model reproduces the measured eigenvalues and eigenvectors while some important structural properties, such as the symmetry, positive definiteness, connectivity, etc., which are very often offered by practical problems, are preserved. Once a model has been updated in such a desirable way, the updated model can be used by practicing engineers with confidence for future design and construction. There are also important applications of model updating in *damage detection* and *health monitoring* in vibrating structures (see the special issue on "Inverse Problems" of the Journal *Mechanical Systems and Signal Processing* [1]. It is important to note that QPEVA and FEMU problems aim at partially reconstructing (modifying) the existing pencil, using knowledge of a few eigenvalues and eigenvectors, rather than constructing it from "de novo". There exist various solutions of QPEVA and FMEU problems and many of them are being used in current engineering practice. Unfortunately, in many cases, the underlying inverse eigenvalue problems have not been investigated in details in the context of developing these solutions. Thus, these techniques often lack rigorous mathematical justifications. *The purpose of this paper is to first give a brief overview of the existing results, including some new ones, on partially prescribed quadratic inverse eigenvalue problems, and explain their roles in existing solution techniques of AVC and FEMU, whenever possible.*

2. Quadratic Eigenvalue Problem

Given three real matrices M, C and K, each of order n, *the quadratic eigenvalue problem (QEP)* is to find scalars λ and vectors ϕ and y such that

$$\begin{aligned} &P(\lambda)\phi = (\lambda^2 M + \lambda C + K)\phi = 0, \\ &y^T P(\lambda) = 0. \end{aligned} \tag{1}$$

The scalars λ are the $2n$ roots of the *characteristic polynomial* $\det P(\lambda) = 0$. The vectors ϕ and y are respectively called *right* and *left eigenvectors* associated with the eigenvalue λ. Unless otherwise stated, t*he right eigenvalue will be referred just as eigenvalue.*

The QEP has always $2n$ eigenvalues (finite and infinite) and up to $2n$ right and left eigenvectors. If M os nonsingular, then all the eigenvalues are finite. The pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ is called the *quadratic matrix pencil* and is conveniently denoted by (M, C, K). We will assume throughout the whole paper that the pencil (M, C, K) is symmetric positive definite, meaning that $M = M^T > 0 \in \mathbb{R}^{n\times n}$, $C = C^T \in \mathbb{R}^{n\times n}$, and $K = K^T \geq 0 \in \mathbb{R}^{n\times n}$.

The pencil is called *regular* if $\det P(\lambda)$ is not identically zero for all values of λ; otherwise it is called *singular. Unless otherwise stated, we will assume that the pencil is regular.*

In mechanical vibration, where the QEP routinely arises, the matrices M, C, and K have special significance. They are known as *mass*, *damping* and *stiffness* matrices. Mechanical systems can be modeled by a system of second order differential equations of the form (usually derived from finite element discretization of a distributed parameter system):

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0.$$

The eigenvalues of the associated quadratic pencil $P(\lambda)$ are related to *natural frequencies* and eigenvectors are called *mode shapes.*

Very often the coefficient matrices have special structures: they are symmetric, M is positive definite (> 0) and is diagonal or tridiagonal, and K is positive semi-definite (≥ 0) and tridiagonal (See Section of Case Studies of this paper). For an account of theory, methods and applications of QEP, see the survey by Tisseur and Meerbergen [2] and the book by Datta [3]. It is important to note that *when the problem is large and sparse, as in the case with most engineering applications, the entire spectrum can not be computed, The state-of-the-art Jacobi-Davidson method [3, 4] is capable to compute only a few extremal eigenvalues and eigenvectors.*

2.1. Orthogonality Relations of the Eigenvectors of Quadratic Matrix Pencil

The following orthogonality relations for the quadratic matrix pencil play an important role in solution of QPEVAP. The relations were derived in [5, 6].

Theorem 1. *[Orthogonality of the Eigenvectors of Quadratic Pencil] Let* $P(\lambda) = \lambda^2 M + \lambda C + K$, *where* $M = M^T > 0$, $C = C^T$, *and* $K = K^T$. *Assume that the eigenvalues* $\lambda_1, ..., \lambda_{2n}$ *are all distinct and different from zero. Let* $\Lambda = diag\ (\lambda_1, \cdots, \lambda_{2n})$ *be the eigenvalue matrix and* $\Phi = (\phi_1, ..., \phi_{2n})$ *be the corresponding matrix of eigenvectors. Then there exist diagonal matrices* D_1, D_2, *and* D_3 *such that*

$$\Lambda \Phi^T M \Phi \Lambda - \Phi^T K \Phi = D_1 \tag{2}$$

$$\Lambda \Phi^T C \Phi \Lambda + \Lambda \Phi^T K \Phi + \Phi^T K \Phi \Lambda = D_2 \tag{3}$$

$$\Lambda \Phi^T M \Phi + \Phi^T M \Phi \Lambda + \Phi^T C \Phi = D_3 \tag{4}$$

and

$$D_1 = D_3 \Lambda;\ D_2 = -D_1 \Lambda;\ D_2 = -D_3 \Lambda^2. \tag{5}$$

Furthermore, if $\{\lambda_1, ..., \lambda_k\}$ *and* $\{\lambda_{k+1}, ..., \lambda_{2n}\}$ *are disjoint, then*

$$\Lambda_1 X_1^T M X_2 \Lambda_2 - X_1 K X_2 = 0, \tag{6}$$

where $\Lambda_1 = diag(\lambda_1, ..., \lambda_k)$, $\Lambda_2 = diag(\lambda_{k+1}, ..., \lambda_{2n})$, *and* X_1 *and* X_2 *are corresponding eigenvector matrices.*

Note: In the linear case $M - \lambda K$, when $M = M^T > 0$ and $K = K^T$, all the eigenvalues are real and it then follows from (4) that $\Phi^T M \Phi$ is a real diagonal matrix. Indeed, the matrix Φ can be scaled so that $\Phi^T M \Phi = I$, which is known as mass-orthogonality relation.

2.2. Real-valued representation for the eigenvalues and eigenvectors of the QEP

The matrix eigenpair (Φ, Λ), is in general complex for the QEP. However, it can be represented as a real matrix eigenpair. In practice, it is more convenient to work with such real-symmetric matrix eigenpairs.

Let $k \leq 2n$, write $\Lambda = diag(\alpha_1 + i\beta_1, \alpha_1 - i\beta_1, ..., \alpha_l + i\beta_l, \alpha_l - i\beta_l, \lambda_{2l+1}, ..., \lambda_k)$ and $\Phi = (u_1 + iv_1, u_1 - iv_1, ..., u_l + iv_l, u_l - iv_l, \phi_{2l+1}, ..., \phi_k)$.

Define $S = diag(S_1, ..., S_l, S_{2l+1}, ..., S_k)$, where

$$S_j = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \ j = 1, ..., l \\ 1, \ j = 2l+1, ..., k \end{cases} \tag{7}$$

Denote $(X, T) = (\Phi S^{-1}, S\Lambda S^{-1})$. Then it is easy to check that X and T are real and

$$X = (u_1, v_1, ..., u_l, v_l, \phi_{2l+1}, ..., \phi_k) \tag{8}$$

$$T = diag(T_1, ..., T_l, T_{2l+1}, ..., T_k), \text{ with} \tag{9}$$
$$T_j = \begin{cases} \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}, \ j = 1, ..., l \\ \lambda_j, \ j = 2l+1, ..., k. \end{cases}$$

Each two by two block of the matrix T corresponds to a complex conjugate pair of eigenvalues of $P(\lambda)$. If the pencil is **semi-simple**, i.e. all the eigenvalues have geometric multiplicity 1, then the matrix

$$col(\Phi, \Phi\Lambda) = \begin{pmatrix} \Phi \\ \Phi\Lambda \end{pmatrix} \text{ is of full rank.} \tag{10}$$

See [7].

Condition (10) will hold if we replace the k-matrix eigenpair (Φ, Λ) by $(\Phi S^{-1}, S\Lambda S^{-1})$.

Definition 1. The pair (X, T), $X \in \mathbb{R}^{k \times 2k}$ and $T \in \mathbb{R}^{2k \times 2k}$, as defined above, is called a k-real matrix eigenpair ($k \leq 2n$) or just a real matrix eigenpair.

The real matrix eigenpair (X, T) satisfies the following matrix equations:

$$MXT^2 + CXT + KX = 0. \tag{11}$$

Assuming that the eigenvalues are semi-simple, we have

$$\begin{pmatrix} X \\ XT \end{pmatrix} \text{ is of full rank.} \tag{12}$$

3. Quadratic Inverse Eigenvalue Problem (QIEP)

Given a partial or complete eigenstructure of a matrix A, the problem of computing the matrix from the given eigenstructure is called the *inverse eigenvalue problem*. Similarly, the inverse eigenvalue problem for the quadratic pencil $P(\lambda)$, denoted by *QIEP*, is to find the matrices M, C and K, such that quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ has a prescribed eigenstructure. Because of their importance in practical applications, both QEP and QIEP are subjects of intensive research nowadays.

An excellent account of the standard inverse eigenvalue problems is given in the books by Chu and Golub [8] and Gladwell [9]. Research on QIEP, specially on structured quadratic inverse eigenvalue problem is still in its infancy and many questions related to these problems remain unanswered. A few results are now available in literature. We will now briefly present some of these recent results and establish their connections with Active Vibration Control (AVC) and Finite Element Model Updating(FEMU) problems.

3.1. Construction from Full Spectrum and Eigenvectors

Construction of the quadratic pencil $P(\lambda)$ from the knowledge of complete spectrum and eigenvectors is rather trivial. For example, if $col(\Phi, \Phi\Lambda)$ is of full-rank and Q is a matrix such that $Qcol(\Phi, \Phi\Lambda) = I$, $Q = (Q_1, Q_2)$, $Q_i \in \mathbb{R}^{2n\times n}$, then the matrices M, C, and K defined by

$$\begin{aligned} &M \text{ – arbitrary} \\ &C = -M\Phi\Lambda^2 Q_2 \\ &K = -M\Phi\Lambda^2 Q_1, \end{aligned} \tag{13}$$

are such that $M\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = 0$.

3.2. Construction from Partial Spectrum and Eigenvectors

From practical applications point of view, it is more desirable to construct the pencil $P(\lambda)$ from the knowledge of only a partial spectrum and associated eigenvectors. This is because, as we have remarked earlier for large and spare models, it is not possible to compute the entire spectrum and the associated eigenvectors. Similarly, due to hardware limitations, only a few eigenvalues and eigenvectors can be measured for a real-life structure or in a vibration laboratory, which correspond to the lowest natural frequencies of a vibration structure.

3.2.1. Unstructured Partial Quadratic Inverse Eigenvalue Problem in Mechanical Vibrations: QPEVAP

It is well-known that vibrating structures, such as air and space crafts, buildings, bridges, etc., sometimes experience dangerous vibrations, caused by resonance, resulting in partial or complete destruction which often leads to loss of human lives and property damage. Infamous incidents, such as the *fall of Tacoma bridge* in USA, *collapse of the Broughton bridge* in England, and recent *wobbling of the Millennium bridge* over River Thames in London, England, and several others, are believed to have been caused by resonance.

A practically effective way to control such vibrations is to apply an active vibration control (AVC) force to the structure using the mechanism of feedback control. Resonance occurs when external periodic force has frequency equal or close to a natural frequency. Thus, AVC amounts to applying control force in such a way that a few eigenvalues of $P(\lambda)$ which correspond to natural frequencies will be eliminated from the structure, while the remaining ones and their corresponding modes (eigenvectors) will be preserved. Mathematically, computation of required feedback matrices leads to quadratic partial eigenvalue assignment problem ($QPEVAP$).

$QPEVAP$: Given a pencil (M, C, K), a part of the spectrum $\{\lambda_i\}_{i=1}^{k}$ with $1 < k << 2n$, a set of suitably chosen eigenvalues $\{\mu\}_{i=1}^{k}$, both closed under complex conjugation, and a control matrix B, find the feedback matrices F and G such that the spectrum of the closed-loop pencil $(M, C - BF^T, K - BG^T)$ consists of $\{\mu_1, ..., \mu_k\}$ and the remaining $2n - k$ eigenvalues of the original pencil (M, C, K) and the corresponding $2n - k$ eigenvectors remain unchanged.

In certain applications, in addition to altering the response of the system, it is desirable to change the shape of the response as well. The shape of the response is determined by the eigenvectors of the quadratic pencil. This consideration gives rise to the following, more general problem; Quadratic Partial Eigenstructure Assignment Problem (QPEAP).

$QPEAP$: Given a pencil (M, C, K), a set $\{\mu_1, ..., \mu_k\}$, and set of vectors $\{y_1, ..., y_k\}$, both closed under complex conjugation, and a control matrix B, find matrices F and G, such that the pencil $(M, C - BF^T, K - BG^T)$ has $\{(\mu_1, y_1)..., (\mu_k, y_k)\}$ as its first k eigenpairs and remaining $2n - k$ eigenpairs are the same as of the original pencil (M, C, K).

A practical solution of QPEVAP was first obtained by Datta, Elhay, and Ram [5], using the quadratic orthogonality relation stated in Theorem 1, in the single input case (that is, when B is a vector). This solution technique was then generalized to the multi-input case by Datta and Sarkissian in

[6] and Ram and Elhay in [10]. The approach proposed in these papers has several desirable practical features. It is *Direct and Partial Modal*. It is "direct", because the problem is solved directly in second-order setting, without transforming it to a standard first-order state-space model, thus avoiding a possible ill-conditioned inversion of the mass matrix and the loss of some of the exploitable properties, very often offered by practical problems, such as definiteness, sparsity, bandness, etc. It is "partial-modal" because the problem is solved using only a few eigenvalues of $P(\lambda)$ that need to be reassigned and the corresponding eigenvectors of.

Most importantly, *the no spill-over property is established by means of mathematically proved results, (see proof of Theorem 2 below) and no model reduction is needed no matter how large the model may be.* This is important from practical applications view points, since in practice it is not possible to verify the no spill-over experimentally or computationally, especially for a large finite element model, which is very often the case in practice.

Theorem 2. *[6] Let matrix B be of full rank. Let the scalars $\{\mu_1, ..., \mu_k\}$ and the eigenvalues of the pencil (M, C, K) be such that sets $\{\lambda_1, ..., \lambda_k\}$, $\{\lambda_{k+1}, ..., \lambda_{2n}\}$ and $\{\mu_1, ..., \mu_k\}$ are disjoint and each set is closed under complex conjugation. Let $Y = (y_1, ..., y_k)$ be the matrix of left eigenvectors associated with eigenvalues $\{\lambda_1, ..., \lambda_p\}$. Let the pair $(P(\lambda), B)$ be partially controllable with respect to $\{\lambda_1, ..., \lambda_k\}$, i.e. $y_i^* B \neq 0, i = 1, ..., k$. Let $\Gamma = (\gamma_1, ..., \gamma_k)$ be a matrix such that*

$$\gamma_j = \overline{\gamma}_i, \text{ whenever } \mu_j = \overline{\mu}_i.$$

Set $\Lambda_1 = diag(\lambda_1, ..., \lambda_k)$ and set $\Sigma = diag(\mu_1, ..., \mu_k)$. Let Z be the unique nonsingular solution of the Sylvester equation

$$\Lambda_1 Z - Z\Sigma = -Y^* B\Gamma,$$

Define the real feedback matrices by

$$F = \Phi Y^* M, \text{ and } G = \Phi(\Lambda_1 Y^* M + Y^* C),$$

where Φ satisfies the linear system $\Phi Z = \Gamma$.

Then matrices F and G are real and the closed-loop pencil $(M, C - BF^T, K - BG^T)$ will have $\{\mu_1, ..., \mu_k, \lambda_{k+1}, ..., \lambda_{2n}\}$ as its eigenvalues and the eigenvectors corresponding to the eigenvalues $\{\lambda_{k+1}, ..., \lambda_{2n}\}$ will remain unchanged.

Proof. We prove here only the last part. The proof of the first part is involved and can be found in [11].

Let $\Lambda_2 = diag(\lambda_{k+1}, ..., \lambda_{2n})$ and X_2 be the corresponding eigenvector matrix. In matrix notations, we then need to prove that

$$MX_2\Lambda_2^2 + (D - BF^T)X_2\Lambda_2 + (K - BG^T)X_2 = 0.$$

The result follows by substituting expressions for F and G into the left-hand side of the equation and noting that (X_2, Λ_2) is a matrix eigenpair of $P(\lambda)$, that is:

$$MX_2\Lambda_2^2 + CX_2\Lambda_2 + KX_2 = 0$$

, and the eigenpairs (Λ_1, X_1) and (Λ_2, X_2) satisfy the orthogonality relation (6).

□

Notes:

- **Robustness in Quadratic Partial Eigenvalue Assignment**. The feedback matrices given by Theorem 2 are parameterized by the matrix Γ, thus yielding a family of feedback matrices for different choices of Γ. Exploiting this parametric nature of the solution, Brahma and Datta [12, 11] and Bai and Datta [13] have recently developed algorithms for minimum feedback norms and minimization of the condition number of the closed-loop eigenvector matrix. *These algorithms constitute a numerically robust solution to QPEVAP.*

- **Solution of QPEAP**. Expressions for feedback matrices for QPEAP, similar to those stated in the theorem above, have been derived by Datta, Elhay, Ram, and Sarkissian [14].

- **An Inverse Eigenvalue Problem**. Solutions of QPEVAP and QPEAP as stated above prompt the following question: *What conditions guarantee the existence of solutions of unstructured partially prescribed quadratic eigenvalue problem with no spill-over*? Note that solutions of QPEVAP and QPEAP required some implicit or explicit mathematical assumptions. *The above quadratic inverse eigenvalue problem has not been solved yet.*

3.3. Structured QIEP with Partially Prescribed Eigenstructure: FEMUP

The solutions of QPEVAP and QPEAP do not require that the structural properties, such as the symmetry, definiteness, etc., remain invariant after application of feedback control. However, there are other practical applications, such as the *finite element model updating*, to be described later in this

paper, for which such requirements must be met. The underlying Partially Prescribed Structured Quadratic Inverse eigenvalue problem (*PSQIEP*) may be stated as follows:

PSQIEP: Given a k-real matrix pair (X,T), $X \in \mathbb{R}^{2n\times k}$ and $T \in \mathbb{R}^{k\times k}$, $k \leq 2n$, what conditions guarantee the existence of the matrices M, C, and K with a *prescribed structure*, such that the pair (X,T) will be a matrix eigenpair of the pencil $\lambda^2 M + \lambda C + K$?

In the following, we state some of the recent results on such structured PSQIEP.

Theorem 3 ([15])**.** *Given pair of matrices (X,T) in the form (8) and (9), $X \in \mathbb{R}^{2n\times k}$, $T \in \mathbb{R}^{k\times k}$, $k \leq n$. Let Q_1 and Q_2 satisfies:*

$$(Q_1\ Q_2)col(X, XT) = 0. \tag{14}$$

Then, the pair (X,T) is an eigenpair of the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$, where symmetric matrices M, C, and K defined by

$$\begin{aligned} M &= Q_2^T Q_2 \\ C &= Q_2^T Q_1 + Q_1^T Q_2 \\ K &= Q_1^T Q_1. \end{aligned} \tag{15}$$

By construction, the matrices M and K are symmetric positive semi-definite. Furthermore, if X is of full column rank, then the resulting pencil (M,C,K) defined by (15) is regular, it is singular if the eigenvalues in T are all distinct and X is not of full rank.

The remaining eigenstructure of the pencil can also be described. See the above paper for details.

A constructive procedure that guarantees that the matrix M is positive definite has been proposed in [16]. Their solution is based on QR decomposition of the eigenvector matrix X.

Theorem 4 ([16])**.** *Given a matrix pair (X,T) in the form (8) and (9), $X \in \mathbb{R}^{2n\times k}$, $\Lambda \in \mathbb{R}^{k\times k}$ with $k \leq n$. Assume that X is of full column rank and T has only simple eigenvalues. Let*

$$X = Q \begin{pmatrix} R \\ 0 \end{pmatrix} \tag{16}$$

be the QR decomposition of X, and let $S = RTR^{-1}$. Then the pair (X,T) is an eigenpair of the symmetric pencil $\lambda^2 M + \lambda C + K$ with M positive definite, where the matrices M, C, and K are defined as follows:

$$M = Q \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} Q^T,\ C = Q \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} Q^T,\ K = Q \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} Q^T, \tag{17}$$

Here

1. $\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ is arbitrarily symmetric positive definite,
2. $C_{22},\ K_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$ are arbitrarily symmetric,
3. $C_{12} \in \mathbb{R}^{(n-k)\times k}$ is arbitrary,
4. $C_{11} = C_{11}^T = -(M_{11}S + S^T M_{11} + R^{-T} D R^{-1}) \in \mathbb{R}^{k\times k}$,
5. $K_{11} = K_{11}^T = S^T M_{11} S + R^{-T} D T R^{-1} \in \mathbb{R}^{k\times k}$,
6. $K_{12} = -(M_{21}S^2 + C_{21}S) \in \mathbb{R}^{(n-k)\times k}$,

with

$$D = diag\left(\begin{pmatrix} \sigma_1 & \eta_1 \\ \eta_1 & -\sigma_1 \end{pmatrix}, ..., \begin{pmatrix} \sigma_l & \eta_l \\ \eta_l & -\sigma_l \end{pmatrix}, \sigma_{2l+1}, ..., \sigma_k \right),$$

and σ's and η's being arbitrary real numbers. Furthermore, there are ways to choose the matrices M_{11}, M_{21} and C_{21} in such a way that resulting matrix K is positive definite. Also, constructions of the real symmetric matrices M, C, and K with M as nonsingular, in case $n \le k \le 2n$, has been considered in [17] and an algorithm has been proposed there.

Note: The above results are capable of producing solutions of PSQIEP only with positive definite or semi-definite matrices M and K. However, as stated before, in most vibration applications, the matrices, M, K and C are not only symmetric and positive definite (semi-definite), they are well-structured too. For example, the matrix M is often diagonal or tridiagonal and the matrix C is tridiagonal. Ram and Elhay [18] earlier presented an algorithm to construct symmetric tridiagonal matrices C and K for a monic quadratic pencil (that is the pencil with $M = I$). Unfortunately, knowledge of complete spectrum and the leading principal minors of orders up to $2n-2$ are needed for their construction. *Construction of the symmetric matrices M, C, and K with a prescribed structure from the knowledge of partial spectrum and eigenvectors, in general, remains a very difficult problem.* For most recent developments, see [19, 20].

4. Finite Element Model Updating

4.1. Mathematical Statement and Engineering and Computational Challenges

Mathematically, Finite Element Model Updating Problem (FEMUP) can be formulated as follows (the subscript a and u stand, respectively, for the analytical and updated vectors and matrices).

FEMUP: Given a structured pencil (M_a, C_a, K_a) and a few of its associated eigenpairs $\{(\lambda_i, \phi_i)\}_{i=1}^k$ with $k << 2n$, assume that newly measured eigenpairs $\{(\mu_i, y_i)\}_{i=1}^k$ have been obtained. Update the pencil (M_a, C_a, K_a) to (M_u, C_u, K_u) such that:

i. The measured data is reproduced by the model; that is, the subset $\{(\lambda_i, \phi_i)\}_{i=1}^k$ is replaced by $\{(\mu_i, y_i)\}_{i=1}^k$.
ii. The structure, such as the symmetry, positive definiteness or semi-definiteness, and sparsity pattern, of the original model is preserved.
iii. There is no spill-over; that is, remaining $2n - k$ eigenvalues and eigenvectors remain unchanged after updating.

There now exists a large amount of model updating methods. Most of the work prior to 1995 is contained in the book by Friswell and Mottershead [21]) and references there in. Some of the more recent results can be found in [22, 23, 24, 25, 26, 27, 28, 29].

There are several engineering and computational challenges associated with the updating problem, which are not explicitly stated in the above definition. These include: (i) dealing with incompleteness of the measured data, and (ii) complex measured data versus real analytical data, and (iii) appearance of spurious modes in the frequency range of interests and (iv) nonorthogonality of measured data. Due to hardware limitations, the measured eigenvectors are very often of much shorter length than the corresponding analytical eigenvectors. However, in order to use these measured data in an updating process, these two sets of data must be of equal length. To remedy this situation, either the order of the analytical model is reduced *(model reduction)* or the measured eigenvectors are expanded *(modal expansion)*. Most of the finite element analysis codes do not include damping and the model is assumed to be symmetric positive definite. Thus the analytical data is very often "real". On the other hand, the measured data from real-life structures are "complex", because all structures have damping. There are several ways to convert "real" data to complex or vice versa. For details of existing procedures of model reduction, modal expansion and conversion of data, see the book by Friswell and Mottershead [21]. One way to make sure that spurious modes will not be introduced into the frequency range of

interests is to insist that the unupdated eigenvalues and eigenvectors remain unchanged. That is, **no spill-over** occurs. Finally, measured data witch comes from a real-life structure very often fails to satisfy the orthogonality relation between the eigenvectors due to measurement errors. Thus, the measured data needs to be updated so that it satisfies the orthogonality relation.

Several recent publications on model updating have addressed some of the above challenges. In the following section the recent work on the subject will be briefly reviewed. We note, however, that structure preserving model updating still remains a difficult task. This is because of lack of existence of supporting mathematical theory for the underlying structure preserving in inverse eigenvalue problems. For the most recent developments see [30] and references there in.

4.2. Methods for Undamped Models

Most existing methods concern with updating of an undamped model and an updating is usually formulated as an optimization problem. There are two types of updating procedures in the literature. The first type, assuming the mass matrix as the reference matrix, update first the measured data so that it satisfies the mass-orthogonality constraint. This is then followed by updating of the stiffness matrix so as to satisfy the constraints on symmetry and reproduction of measured data. A well-known method of this type is due to Baruch and Bar-Itzhack [31]. Their method updates the measured eigenvector matrix first so that it satisfies the mass-orthogonality relation and the mass-weighted norm is minimized. The stiffness matrix K is then updated using this updated eigenvector matrix.

$$\begin{aligned} \textbf{Stage I} \qquad & \frac{1}{2}||M_a(\Phi_m - \Phi_u)|| \rightarrow \min \\ \text{s.t.} \qquad & \Phi_u M_a \Phi_u = I. \end{aligned}$$

$$\begin{aligned} \textbf{Stage II} \qquad & \frac{1}{2}||M_a^{-1/2}(K_a - K_u)M_a^{-1/2}|| \rightarrow \min \\ \text{s.t.} \qquad & K_u\Phi_u - M_a\Phi_u\Lambda_m = 0 \\ & K_u = K_u^T. \end{aligned}$$

The other methods update, either separately or simultaneously, the mass and stiffness matrices, satisfying the mass-orthogonality, symmetry and eigenvalue-eigenvector constraint. [21, 28, 32, 33].

4.2.1. Undamped Model Updating with no Spurious Modes and with Incomplete Measured Data

The existing model updating methods, including the above method of Baruch-Itzhack are not capable of preventing the appearance of the spill-over in the updated model. Also, as said before that a standard practice to deal with the difficulty of incomplete measured data is to reduce the order of the model or expand the measured data. Below we present a method for undamped model that addresses these piratical issues. The method was initially proposed in the Ph.D dissertation of Carvalho [34] and later was fully developed and applied to some practical problems in [22]. Specifically, *the method retains the eigenvalues and eigenvectors of the original pencil which are not updated and reconstructs the missing degrees of freedom in the measured eigenvectors in such a way that updated eigenvector matrix satisfies the mass-orthogonality relation.*

Conceder updating the pencil $\lambda^2 M + K = 0$, where M and K are symmetric positive semi-definite. Therefore, every eigenvector x is real and corresponds to two purely imaginary eigenvalues $\lambda = \pm i\alpha$. We can have a *compact representation* (X, Λ) of the finite eigenstructure of $P(\lambda)$ satisfying $MX\Lambda^2 + KX = 0$, where $X \in \mathbb{R}^{n\times n}$ and $\Lambda \in \mathbb{C}^{n\times n}$ ($\Lambda^2 \in \mathbb{R}^{n\times n}$ is a diagonal matrix with non-positive entries).

Let $X = (X_1, X_2)$ and $\Lambda = diag(\Lambda_1, \Lambda_2)$, where Λ_1 is $k \times k$ and X_1 is a corresponding partitioning of X. Let $\{\mu_1, ..., \mu_k\}$ and $\{y_1, ..., y_k\}$ be the sets of k ($k << 2n$) eigenvalues and eigenvectors measured from an experimental structure. Let $Y = (y_1, ..., y_k)$ and $\Sigma = diag(\mu_1, ..., \mu_k)$. Let

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix},\ Y_1 \in R^{m\times m},\ Y_2 \in R^{(n-m)\times m} \tag{18}$$

and assume that only Y_1 is known. The question is : *how we can construct Y_2 and then update the matrix K to K_u, such that*

$$(Y,\ X_2),\ \begin{pmatrix} \Sigma & 0 \\ 0 & \Lambda_2 \end{pmatrix}$$

is the matrix eigenpair of the updated model? In other words, the updated model should reproduce measured eigenvectors and eigenvalues and the rest of the eigenstructure remains unchanged. The following orthogonality relations, proved in [22], play the central role in the proposed updating procedure.

Theorem 5. *Let $P(\lambda) = \lambda^2 M + K$ be a be a symmetric pencil with M positive definite and K semi-definite, with distinct nonzero eigenvalues. Let*

(X, Λ) *be the matrix eigenpair of this pencil. Then the matrices* D_1 *and* D_2 *defined by*

$$D_1 = X^T M X, D_2 = X^T K X$$

are diagonal and $D_2 = -D_1\Lambda^2$.

(b) Furthermore, suppose that Λ_1 *and* Λ_2 *do not have a common eigenvalue, then*

$$X_1^T M X_2 = 0, \text{ and } X_1^T K X_2 = 0. \tag{19}$$

Theorem 6 (*Eigenstructure Preserving Updating*)**.** *Assume that* Λ_1 *and* Λ_2 *do not have a common eigenvalue. Then, for every symmetric* $k \times k$ *matrix* Ψ*, the updated pencil* $P_u(\lambda) = \lambda^2 M + K_u$*, where*

$$K_u = K - M X_1 \Psi X_1^T M \tag{20}$$

is such that

$$M X_2 \Lambda_2^2 + K_u X_2 = 0. \tag{21}$$

That is, the eigenvalues and eigenvectors of the original finite element model which are not to be affected by updating, remain unchanged.

Proof. Substituting the expression of K_u from (20) into (21) and using the orthogonality relation (19) above, we have

$$M X_2 \Lambda_2^2 + K_u X_2 = M X_2 \Lambda_2^2 + K X_2 - M X_1 \Psi X_1^T M X_2 = 0$$

□

The question now is how to choose the matrix Ψ such that it is symmetric and the measured eigenvalues and eigenvectors will be reproduced by the updated model? That is, for what choice of Ψ, the matrix K_u is symmetric and and the following eigenvalue-eigenvector relation holds :

$$M Y \Sigma^2 + K_u Y = 0?$$

The following result shows how to do so.

Theorem 7. *Suppose that the matrix* Y *s.t.* $Y^T M Y$ *is a diagonal matrix. Let* Ψ *satisfy the Sylvester matrix equation:*

$$(Y^T M X_1) \Psi (Y^T M X_1)^T = Y^T M Y \Sigma^2 + Y^T K Y, \tag{22}$$

then Ψ *is symmetric and*

$$M Y \Sigma^2 + K_u Y = 0, \text{where } K_u \text{ as in (20)} \tag{23}$$

The above result requires the existence of a solution of the Sylvester equation (22). The following alternative condition has been recently been established by Chu et al [35].

Theorem 8. *There exists matrix a symmetric matrix $\Psi \in R^{k\times k}$, such that both (20) and (23) are satisfied, if and only if*

$$Y = X_1 V D,$$

for some orthogonal matrix $V \in R^{k\times k}$ and some nonsingular diagonal matrix $D \in R^{k\times k}$.

It now remains to show how to find the missing part of Y_2 of Y so that Y is mass normalized. Let

$$MX_1 = (U_1\ U_2)\begin{pmatrix} Z \\ 0 \end{pmatrix}$$

be the QR decomposition of MX_1.

Let $M = (M_1, M_2)$ and $K = (K_1, K_2)$ be conformable partitionings of M and K, where $M_1, K_1 \in R^{n\times k}$ and $M_2, K_2 \in R^{n\times(n-k)}$. Then (23) becomes

$$\begin{pmatrix} U_1^T \\ U_2^T \end{pmatrix}(M_1\ M_2)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}\Sigma^2 = (K_1\ K_2)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} Z \\ 0 \end{pmatrix}\Psi X_1^T MY,$$

Compute now Y_2 by solving the Sylvester equation:

$$U_2^T M_2 Y_2 \Sigma^2 + U_2^T K_2 Y_2 = -U_2^T(K_1 Y_1 + M_2 Y_1 \Sigma_1^2). \tag{24}$$

The it is easy to see that $Y = col(Y_1, Y_2)$ is such that $Y^T MY$ is a diagonal matrix.

The above discussions leads to the following algorithm. Details of the proof could be found in [22].

4.3. Model Updating for Damped Models

Updating of a damped model is rarely considered in the engineering literature. Earlier, several control-theoretic methods for updating were developed [36, 37, 38]. Unfortunately, the use of control destroys the symmetry. There are optimization based techniques which are capable of preserving the symmetry. Friswell, et al [26] gave an explicit formula for updating a damped model, and a numerical algorithm based on the formula was developed by Kuo et al [39].

In this section, we discuss two recent updating schemes for damped models.

Algorithm 1 Model Updating algorithm for an undamped model with guaranteed no spill-over

INPUT: $M = M^T > 0$, $K = K^T$, $\Sigma = diag(\mu_1, ..., \mu_k)$, Y_1
OUTPUT: Updated stiffness matrix K_u and Y_2 such that (23) is satisfied and $Y = col(Y_1, Y_2)$ is such that $Y^T M Y$ is a diagonal matrix

1: Compute Y_2 by solving Equation (24) and form the matrix $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$.
2: Orthogonalize matrix Y, by computing LDL decomposition of $Y^T M Y = LDL^T$. Update the matrix Y by $Y \leftarrow YL^{-T}$
3: Compute Ψ by solving the following algebraic system of equation:

$$(Y^T M X_1)\Psi(Y^T M X_1)^T = Y^T M Y \Sigma^2 + Y^T K Y,$$

4: Update the stiffness matrix

$$K_u = K - M X_1 \Psi X_1^T M.$$

4.3.1. An Eigenvalue Embedding Method

Carvalho et al [40] developed an updating scheme which preserves symmetry, and uses low-rank updating, and guarantees the *no spill-over* with mathematical proof. Unfortunately, in their work, only updating of measured eigenvalues were considered. The scheme did not take into account of updating of the measured eigenvectors.

We illustrate the scheme by means of the following theorem, where a real isolated eigenvalue is updated. But the process is more general and is capable of updating both real and complex eigenvalues.

Theorem 9. *Let (λ, x) be a real isolated eigenpair of $P(\lambda) = \lambda^2 M + \lambda C + K$ with $x^T K x = 1$. Let λ be reassigned to μ. Assume that $1 - \lambda\mu\theta \neq 0$ and $1 - \lambda^2\theta \neq 0$, where $\theta = x^T M x$ and $\epsilon = (\lambda - \mu)/(1 - \lambda\mu\theta)$. Then the following updated quadratic matrix pencil*

$$P_u(\lambda) = \lambda^2 M_u + \lambda C_u + K_u$$

with

$$M_u = M - \epsilon\lambda M x x^T M$$

$$C_u = C + \epsilon(M x x^T K + K x x^T M)$$

$$K_u = K - \frac{\epsilon}{\lambda} K x x^T K$$

is such that

i. The eigenvalues of $P_u(\lambda)$ are the same as those of $P(\lambda)$ except that λ has been replaced by μ (assignment of real eigenvalues).

ii. x is also an eigenvector of $P_u(\lambda)$ corresponding to the embedded eigenvalue μ.

iii. If (λ_2, x_2) is an eigenpair of $P(\lambda)$, where $\lambda_2 \neq \lambda$, then (λ_2, x_2) is also an eigenpair of $P_u(\lambda)$ (No Spill-over).

A generalization of the result was obtained by Lancaster in [41, 42]. In his work, Lancaster considers Hermitian matrices (instead of real symmetric). The method proposed by Lancaster is also capable of updating measured eigenvectors. Let (X, Λ) be the matrix eigenpair of $P(\lambda)$, where X and Λ are in the following form

$$\Lambda = \begin{pmatrix} U_1 + iW & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & U_1 - iW \end{pmatrix}, \; X = (X_c, X_{R1}, X_{R2}, \overline{X}_c). \quad (25)$$

Here $X_c \in R^{n\times(n-r)}$ is matrix of complex eigenvectors and matrix $U_1 + iW$ is the matrix of corresponding eigenvalues. Matrices $X_{R1}, X_{R2} \in R^{n\times r}$ are matrices of real eigenvectors which correspond to eigenvalue matrices U_2, U_3.

Let P be a permutation matrix, defined by

$$P = \begin{pmatrix} 0 & 0 & 0 & I_{n-r} \\ 0 & I_r & 0 & 0 \\ 0 & 0 & -I_r & 0 \\ I_{n-r} & 0 & 0 & 0 \end{pmatrix}. \quad (26)$$

Consider now the following partitioning of the spectral data matrices and the matrix P:

$$X = (X_1, \; X_2), \; \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \; P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

Let, Λ_1 is to be modified to Σ and Λ_2 is its unknown compliment and stays unchanged after updating. The eigenvectors in X_1 associated with Λ_1 are to be replaced by Y.

Theorem 10. *Let (M, C, K) be a Hermitian quadratic matrix pencil and matrix eigenpair (X, Λ) be of the form defined above in (25). Let matrix P be defined as in (26). Given Σ, a matrix of the same size as Λ_1 and Y, a matrix of the same size as X_1. Assume that Σ has the same number of real eigenvalues and complex conjugate pairs of eigenvalues and if $\sigma_i = \overline{\sigma_j}$, then $y_i = \overline{y}_j$. Define*

$$\begin{aligned} M_u &= \left(M^{-1} + (\overline{S}_1 - S_1)\right)^{-1} && (27)\\ C_u &= M_u \left(M^{-1}CM^{-1} - (\overline{T}_1 - T_1)\right) M_u && (28)\\ K_u &= -M_u \left(M^{-1}(CM^{-1}C - K\right) M^{-1} + (\overline{U}_1 - U_1)M_U + C_u M_u^{-1} C_u && (29) \end{aligned}$$

where

$$\begin{aligned} S_1 = X_1\Lambda_1 P_1 X_1^*,\ \overline{S}_1 = Y\Sigma P_1 Y^* \\ T_1 = X_1\Lambda_1^2 P_1 X_1^*,\ \overline{T}_1 = Y\Sigma^2 P_1 Y^* \\ U_1 = X_1\Lambda_1^3 P_1 X_1^*,\ \overline{U}_1 = Y\Sigma^3 P_1 X_1^*. \end{aligned}$$

Then

$$X = (Y,\ X_2),\ \Lambda = \begin{pmatrix} \Sigma & 0 \\ 0 & \Lambda_2 \end{pmatrix}$$

is the matrix eigenpair of the quadratic pencil (M_u, C_u, K_u).

This approach admits only changes from real to real eigenvalues and complex conjugate pairs to complex conjugate pairs. Thus, matrix P remains unchanged. Also note that the formulas for (M_u, C_u, K_u) do not require the knowledge of the (X_2, Λ_2). For details see Lancaster [42].

4.4. Quadratic Model Updating with Measured Data Satisfying Orthogonally Relations

In this section, we describe a recently proposed method by Datta, et al. [43] for damped model updating that preserves symmetry and the measured data satisfies a quadratic orthogonality constraint. The role of the orthogonality constraint is explained by proving an inverse eigenvalue theorem for PSQIEP.

4.4.1. Existence of Symmetric Solution of the Model Updating Problem

The following result gives a necessary and sufficient conditions for the existence of a symmetric matrix K while other two matrices M and C remain fixed. However, other variations are also possible. Indeed, Cai et al [17] have investigated the case where all three matrices M, C, and K are to be constructed.

Theorem 11. *Given $M = M^T \in \mathbb{R}^{n\times n}$, $C = C^T \in \mathbb{R}^{n\times n}$; and $T \in \mathbb{R}^{k\times k}$, $X \in \mathbb{R}^{n\times k}$, matrices of the form (9), (8), respectively. Let (X, T) satisfy condition (12) and all the eigenvalues of T are distinct. Then there is a real symmetric matrix K such that $MXT^2 + CXT + KX = 0$ if and only if*

$$\begin{pmatrix} X \\ XT \end{pmatrix}^T \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} X \\ XT \end{pmatrix} = B(X, T), \tag{30}$$

where B is some block-diagonal matrix with blocks of the following form

$$B = diag(B_1, ..., B_l, B_{2l+1}, ..., B_k),\ B_j = \begin{cases} \begin{pmatrix} a_j & b_j \\ b_j & -a_j \end{pmatrix},\ j = 1, ..., l \\ b_j,\ j = 2l+1, ..., k, \end{cases} \tag{31}$$

and $BT = T^T B$.

Proof. Assume M, C and K are given and The matrix eigenpair (X, T) satisfies the relation (11), which can be written as

$$\begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} X \\ XT \end{pmatrix} = \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} X \\ XT \end{pmatrix} T. \tag{32}$$

By multiplying the above relation by

$$\begin{pmatrix} X \\ XT \end{pmatrix}$$

we obtain

$$\begin{pmatrix} X \\ XT \end{pmatrix}^T \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} X \\ XT \end{pmatrix} = B(X, T), \tag{33}$$

and

$$\begin{pmatrix} X \\ XT \end{pmatrix}^T \begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} X \\ XT \end{pmatrix} = B(X, T), \tag{34}$$

Since M, C, K are symmetric, it is also easy to show that $BT = T^T B$

The form (31) of matrix B follows from the relation :

$$B = S^{-T} D_3 S^{-1},$$

where the matrix D_3 as defined in (4), and S is defined in (7).

□

We now show how to find a symmetric K which will satisfy the eigenvalue-eigenvector relation (11) for given $M = M^T, C = C^T$. First note that since (X, T) satisfies the full-rank condition (12), we can find an extension of the matrices X and T, $X_{ext} = [X\ \hat{X}] \in \mathbb{R}^{n\times 2n}$, $T_{ext} = diag(T, \hat{T}) \in \mathbb{R}^{2n\times 2n}$, such that

$$\begin{pmatrix} X_{ext} \\ X_{ext}T_{ext} \end{pmatrix} \text{is of full rank and}$$

$$\begin{pmatrix} X_{ext} \\ X_{ext}T_{ext} \end{pmatrix}^T \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} X_{ext} \\ X_{ext}T_{ext} \end{pmatrix} = [B(X,T), \hat{B}] = B_{ext}(X_{ext}, T_{ext}).$$

Here $\hat{T}, \hat{B}$ are real block-diagonal matrices. Then we can take K as the solution to the following linear system

$$\begin{pmatrix} X_{ext} \\ X_{ext}T_{ext} \end{pmatrix}^T \begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} X_{ext} \\ X_{ext}T_{ext} \end{pmatrix} = B_{ext}T_{ext}$$

i.e. $K = X_{ext}^{-T}(T_{ext}^T X_{ext}^T M X_{ext} T_{ext} - B_{ext}T_{ext})X_{ext}^{-1}$. This is a real symmetric matrix (note, that $B_{ext}T_{ext}$ is a symmetric matrix).

A special case: the symmetric positive definite linear pencil $K - \lambda M$. Given $M = M^T > 0$ and $\Lambda = diag(\lambda_1, ..., \lambda_n)$ and $X = (x_1, ..., x_n)$, where $\lambda_i \in (R)$ and $x_i \in \mathbb{R}^n$. Then there exists a symmetric K such that $K\Phi = M\Phi\Lambda$, if and only if $\Phi^T M \Phi = I$.

Remark: The above result has bees used implicitly or explicitly in most of the model updating methods, including the method of Baruch and Bar-Itzhack without prior mathematical justification.

4.4.2. A Two-Stage Model Updating Scheme

In this section, we introduce our two-stage model updating scheme for FEMU in an optimization setting. Let (X_M, T_M) denote the measure matrix eigenpair.

Stage I: **Updating of Measured Eigenvector Matrix Satisfying the Quadratic Orthogonality Constraint**:

$$(\mathcal{P}) \qquad \begin{aligned} \min \quad & \frac{1}{2}||W_1(X - X_M)||_F^2 \\ \text{s.t.} \quad & H(X) = 0 \\ & X \in \mathbb{R}^{n\times k}, \end{aligned}$$

where

$$H(X) = B(X, T_M)T_M - T_M^T B(X, T_M) \tag{35}$$

with $B(X, T_M)$ of the form (34) and W_1 is some positive-definite weighting matrix.

Stage II: **Updating the Stiffness Matrix Satisfying Eigenvalue-Eigenvector Constraint and Symmetry**: Let X_u be a solution from Stage I. Then

$$\begin{aligned} (\mathcal{Q}) \qquad \min \quad & \frac{1}{2}||W_2(K - K_a)W_2||_F^2 \\ \text{s.t.} \quad & M_a X_u T_M^2 + C_a X_u T_M + K X_u = 0 \\ & K^T = K \quad \text{(symmetry)} \\ & K \in \mathbb{R}^{n\times n}, \end{aligned}$$

where $W_2 > 0$ is a positive-definite weighting matrix. The solution to the problem will be denoted by K_u.

Remarks: It is also possible to update both the stiffness and damping matrices satisfying the orthogonality relation of Stage I. This will require reformulation of the problem. Such reformulation is currently being investigated.

4.5. A Solution Method and its Convergence Properties

Since the Stage-2 problem is a convex quadratic problem, it can be solved by applying excellent numerical methods that are available in the literature [44].

However, in our numerical experiment, we have used the same method to solve the Stage II problem that we have developed for Stage I.

We assume that the weighting matrices are simply identity matrices.Set

$$f(X) = \frac{1}{2}||X - X_M||_F^2.$$

Then the Lagrangian function for $(\mathcal{P})$ is:

$$L(X, Y) = f(X) + \langle Y, H(X)\rangle$$

where $Y \in \mathbb{R}^{k\times k}$. Some remarks about our Lagrange function L are in order. By definition, $H(X)^T = -H(X)$. Hence the system $H(X) = 0$ defines $k(k-1)/2$ constraints.

The necessary optimality conditions for $(\mathcal{P})$ can now be stated as follows: find a pair (X_*, Y_*) such that

$$\nabla_X L(X_*, Y_*) = 0, \quad H(X_*) = 0. \tag{36}$$

Problem $(\mathcal{P})$ is a polynomial programming problem. But the feasible region defined by the polynomial equality constraints are nonconvex in general. Hence we are dealing with a nonconvex minimization problem with equality constraints.

Generally speaking, if X_* is an optimal solution for $(\mathcal{P})$, and the constraint system satisfies certain regularity condition at X_*, then there is a $Y_* \in \mathbb{R}^{k\times k}$ such that (X_*, Y_*) satisfies (36). Elements of Y_* are usually called Lagrangian multipliers.

We present here an augmented Lagrangian method to solve this problem. The literature on augmented Lagrangian methods is vast. We refer the reader to [45, 46] for a thorough treatment on this class of methods and its convergence theory. To this end, we introduce the following parameterized family of the augmented Lagrangian functions:

$$L_\rho(X, Y) = L(X, Y) + \frac{\rho}{2}||H(X)||_F^2, \tag{37}$$

where ρ is a positive constant.

Algorithm 2 The augmented Lagrangian method for $(\mathcal{P})$

INPUT: $X_0, Y_0, \rho_0 > 0, 0 < \beta < 1$, and $\epsilon > 0$

OUTPUT: Solution to $(\mathcal{P})$

1: **for** i = 0,1,... **do**

2: Stop if $||\nabla_X L(X_i, Y_i)|| \le \epsilon$, and $||H(X_i)|| \le \epsilon$

3: Solve the unconstrained optimization subproblem:

$$(\mathcal{P}(i)) \qquad \min_X \; L_{\rho_i}(X, Y_i)$$

$$\text{with the stoping criteria, } ||\nabla_X L_{\rho_i}(X, Y_i)||_F < \beta^i$$

Let X_{i+1} be the solution of $(\mathcal{P}(\text{i}))$

4: Update the multiplier matrix:

$$Y_{i+1} = Y_i + \rho_i H(X_{i+1}).$$

Then choose $\rho_{i+1} > \rho_i$

5: **end for**

The existence of a global minimizer in Step 3: for $\mathcal{P}(i)$, the following proposition guarantees that there is a solution.

Proposition 12. *Let $\rho > 0$ and $Y \in \mathbb{R}^{k\times k}$. Then $argmin_X L_\rho(X, Y)$ is non-empty.*

The convergence of the proposed method: Finding a global solution to a nonconvex optimization problem is a subject of intensive research nowadays. It is challenging and remains a difficult task. In [43], Step 3 problem in the above algorithm was solved by finding solutions to a sequence of critical points instead. For details, we refer the readers to the above paper.

4.5.1. Computation of the Gradient Formulas

Computing the gradient formulas is rather a trivial task. However, in our case challenge is that these gradient formulas are to be computed using the knowledge of a few measured eigenvalues, eigenvectors, and the corresponding analytical eigenvalues. This is particularly important since our numerical experiments are conducted in MATLAB environment.

The basic idea for deriving gradient formulas comes from operator theory on adjoint operators as has been used in [47].

4.6. Gradient Formula for Stage I Problem

It is easy and elementary to see that

$$\nabla f(X) = X - X_M.$$

Let

$$\begin{aligned} h(X) &= 1/2||H(X)||_F^2 = 1/2\langle H(X), H(X)\rangle \\ g_Y(X) &= \langle Y, H(X)\rangle. \end{aligned}$$

Then

$$\nabla h(X) = 2(M_a X G(X) {T_M}^T + (C_a X + M_a X T_M) G(X)), \tag{38}$$

where $G(X) = H(X){T_M}^T - T_M H(X)$ and $H(X)$ is given by (35);

$$\nabla g_Y(X) = 2(M_a X (Y T_M^T - T_M Y) T_M^T + (C_a X + M_a X T_M)(Y T_M^T - T_M Y)). \tag{39}$$

The details of the derivation of formula (38) are given in the paper.

With the above gradient formulas, the gradient formulas for L and L_ρ with respect to X can be written as:

$$\begin{aligned}\nabla_X L(X,Y) &= \nabla f(X) + \nabla g_Y(X),\\ \nabla_X L_\rho(X,Y) &= \nabla_X L(X,Y) + \rho \nabla h(X).\end{aligned}$$

4.6.1. Gradient Formulas for Problem $\mathcal{Q}$

Gradient Functions in Problem $(\mathcal{Q})$ are much simpler, and can be written down as follows. For $\bar{h}(K) = 1/2||K - K_a||_F^2$, we have

$$\nabla \bar{h}(K) = K - K_a,$$

For $\bar{f}(K) = 1/2||M_a X_u T_M^2 + C_a X_u T_M + K X_u||_F^2$, we have

$$\nabla \bar{f}(K) = (M_a X_u T_M^2 + C_a X_u T_M + K X_u) X_u^T.$$

We will now use the above gradient formulas to obtain the necessary optimality conditions for Stage I.

Necessary optimality conditions in matrix form for (P):

Find X and Y such that

$$\begin{aligned}\nabla f(X) + \nabla g_Y(X) &= 0,\\ H(X) &= 0,\end{aligned}$$

where $H(X)$is given by (35), and $\nabla g_Y(X)$ is given by (39).

The above necessary optimality conditions expressed in terms of given matrices are significant. It not only opens up the possibility of solving $(\mathcal{P})$ by solving the above systems of equations (such as by Newton's method for nonlinear equations), but also forms the basis for sensitivity analysis when the problem data undergoes small changes.

4.6.2. Case Studies

In this section, we present the results of our numerical experiments on

- A spring-mass system of 10 Degree of Freedom (DoF) [21]
- A vibrating beam.

The data for our experiments are set up as follows:

- The matrices M_a, C_a are kept fixed.
- To simulate the measured data (X_M, T_M), we add a random noise with some Gaussian distribution to the eigendata of the analytical model.

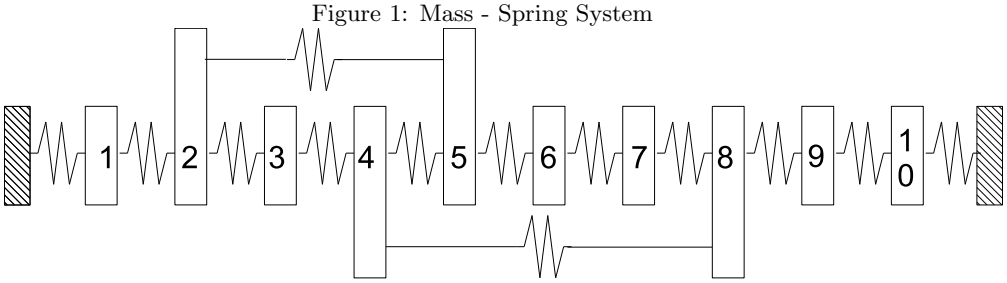
- The weighting matrices were taken as identity matrices.

We used MATLAB with double arithmetics to run numerical experiments. As a solver of an unconstrained optimization problem, MATLAB optimization toolbox routine **fminunc** which implements the well-known BFGS Quasi-Newton method [44] has been used.

4.6.3. A mass-spring system of 10 DoF

Consider the example of a mass-spring system of 10 DoF, as depicted in Figure 1. In this example all rigid bodies have a mass of 1 kg, and all springs have stiffness 1 kN/m. The analytical model is given by:

Figure 1: Mass - Spring System



$$M_a = I$$

$$C_a = \begin{pmatrix} 0.48 & -8.3 & & & & & & & & \\ -8.38 & 8.38 & -1.02 & & & & & & & \\ & -1.0 & 1.02 & -7.28 & & & & & & \\ & & -7.28 & 7.28 & -4.40 & & & & & \\ & & & -4.40 & 4.40 & -9.97 & & & & \\ & & & & -9.97 & 9.97 & -5.62 & & & \\ & & & & & -5.62 & 5.62 & -4.65 & & \\ & & & & & & -4.65 & 4.65 & -4.19 & \\ & & & & & & & -4.19 & 4.19 & -2.11 \\ & & & & & & & & -2.11 & 2.11 \end{pmatrix}$$

$$K_a = \begin{pmatrix} 2000 & -1000 & & & & & & & & \\ -1000 & 3000 & -1000 & & -1000 & & & & & \\ & -1000 & 2000 & -1000 & & & & & & \\ & & -1000 & 3000 & -1000 & & & -1000 & & \\ & -1000 & & -1000 & 3000 & -1000 & & & & \\ & & & & -1000 & 2000 & -1000 & & & \\ & & & & & -1000 & 2000 & -1000 & & \\ & & & -1000 & & & -1000 & 3000 & -1000 & \\ & & & & & & & -1000 & 2000 & -1000 \\ & & & & & & & & -1000 & 2000 \end{pmatrix}$$

The measured data for our experiment was simulated by reducing stiffness of the spring between masses 2 and 5 to 600 N/m and adding Gaussian noise with $\sigma = 2\%$.

The analytical eigenvalue and eigenvector matrices are:

$$T_a = \begin{pmatrix} -6.23 & 71.1 & 0 & 0 \\ -71.1 & -6.23 & 0 & 0 \\ 0 & 0 & -3.67 & 65.9 \\ 0 & 0 & -65.9 & -3.67 \end{pmatrix}, X_a = \begin{pmatrix} 0.142 & 0.001 & -0.161 & -0.001 \\ -0.438 & 0.020 & 0.372 & -0.050 \\ 0.288 & 0.065 & -0.056 & 0.031 \\ -0.502 & -0.206 & -0.191 & 0.087 \\ 0.479 & 0.148 & -0.296 & -0.034 \\ -0.136 & -0.011 & 0.263 & 0.091 \\ -0.066 & 0.011 & -0.346 & -0.145 \\ 0.339 & 0.003 & 0.599 & 0.063 \\ -0.122 & -0.007 & -0.296 & -0.093 \\ 0.040 & 0.010 & 0.115 & 0.075 \end{pmatrix}$$

The measured eigenvalue and eigenvector matrices are:

$$T_M = \begin{pmatrix} \text{-6.16} & 69.8 & 0 & 0 \\ \text{-69.8} & \text{-6.16} & 0 & 0 \\ 0 & 0 & \text{-4.7} & 64.9 \\ 0 & 0 & \text{-64.9} & \text{-4.7} \end{pmatrix}, X_M = \begin{pmatrix} 0.102 & 0.026 & \text{-0.172} & \text{-0.023} \\ \text{-0.283} & \text{-0.061} & 0.401 & 0.023 \\ 0.282 & 0.115 & \text{-0.195} & \text{-0.005} \\ \text{-0.579} & \text{-0.240} & 0.074 & 0.068 \\ 0.341 & 0.242 & \text{-0.354} & \text{-0.202} \\ \text{-0.067} & \text{-0.054} & 0.286 & 0.242 \\ \text{-0.168} & 0.036 & \text{-0.249} & \text{-0.340} \\ 0.508 & \text{-0.042} & 0.362 & 0.382 \\ \text{-0.207} & 0.009 & \text{-0.183} & \text{-0.246} \\ 0.077 & 0.011 & 0.060 & 0.130 \end{pmatrix}$$

Results of Stage I

Orthogonality condition was not satisfied with the measured eigenvector matrix as shown by the residual:

$$||H(X_M)||_F = 17.7$$

However, with preprocessing of the measured data using Stage I we obtained

$$X_u = \begin{pmatrix} 0.086 & 0.128 & 0.139 & 0.021 \\ \text{-0.270} & \text{-0.226} & \text{-0.324} & 0.055 \\ 0.266 & 0.176 & 0.233 & \text{-0.131} \\ \text{-0.437} & \text{-0.239} & \text{-0.222} & 0.264 \\ 0.227 & 0.387 & 0.343 & 0.041 \\ 0.031 & \text{-0.261} & \text{-0.218} & \text{-0.232} \\ \text{-0.147} & 0.205 & 0.183 & 0.312 \\ 0.395 & \text{-0.239} & \text{-0.226} & \text{-0.544} \\ \text{-0.145} & 0.142 & 0.149 & 0.244 \\ 0.051 & \text{-0.061} & \text{-0.064} & \text{-0.096} \end{pmatrix}$$

The *orthogonality condition is now satisfied with this updated eigenvector matrix* as shown by the following residual:

$$||H(X_u)||_F = 1.97 \times 10^{-8}.$$

Results of Stage II

With the updated eigenvector matrix X_u from Stage I:

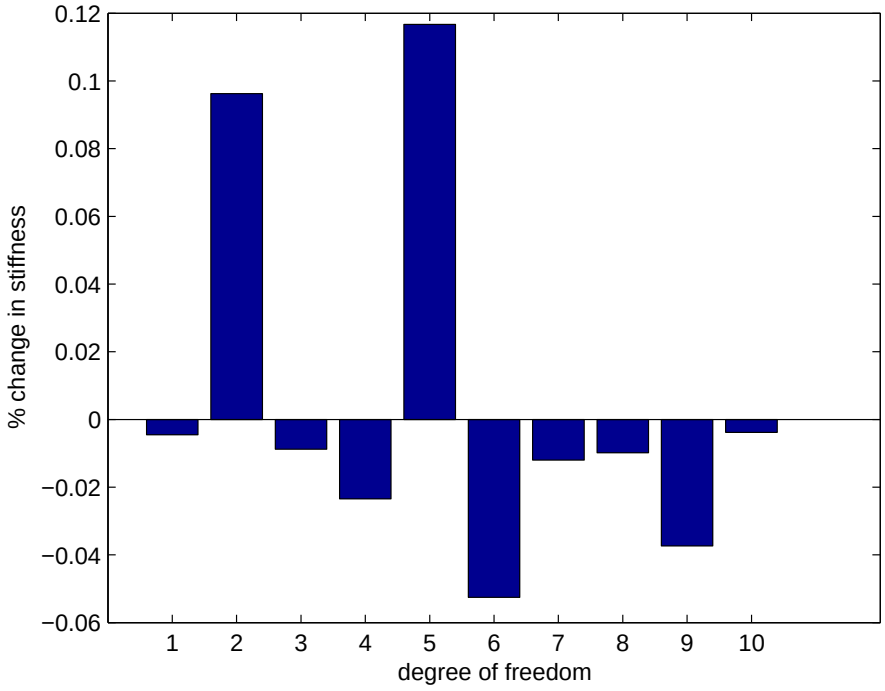
- The updated matrix K_u was symmetric, as shown by the residual norm:
$$||K_u - K_u^T||_F = 9.293 \times 10^{-9}.$$

- *The measured eigenvalues and corrected measured eigenvectors were reproduced accurately by the updated model*, as shown by the following residual:

$$||R(K_u)||_F = 1.78143 \times 10^{-6},$$

where $R(K) = M_a X_u T_M^2 + C_a X_u T_M + K X_u$.

Figure 2: The percentage change in the diagonal elements of the stiffness matrix for mass-spring system



Note: It is clear from the above figure that the largest changes correspond to degrees of freedom 2 and 5. The changes corresponding to the other degrees of freedom are reasonably small.

4.6.4. Vibrating Beam

Consider a discrete spring-mass model of a vibrating beam [48], which consists of $n + 2$ masses $\{m_i\}_{i=-1}^{n}$, linked by massless rigid rods of length $\{l_i\}_{i=0}^{n}$ which are themselves connected by n rotational springs of stiffness

$\{k_i\}_{i=1}^n$. This model corresponds to a finite difference approximation of a beam with distributed parameters. The vibration of the beam with clamped left hand end and with no force applied at the free end is governed by:

$$M\ddot{x} + Kx = 0,$$

where

$$K = EL^{-1}E\hat{K}E^TL^{-1}E^T$$

$\hat{K} = diag(k_1, \ldots, k_n)$, $L = diag(l_1, \ldots, l_n)$, $M = diag(m_1, \ldots, m_n)$,

$$E = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ldots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \ldots & 0 & 1 & -1 \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix}$$

The simulated beam has 16 rods of length 1/16 m each, and all masses are 0.1 kg.

Results of Stage I

The measured data was obtained from the analytical data in the same way as the previous example. To simulate the measured data, the coefficients k_3, k_5, k_9 were reduced by 40%, 50%, 30%, respectively and gaussian noises with $\sigma = 2\%$ were added. The simulated measured eigenvector matrix became:

$$X_M = \begin{pmatrix} -1.2528 \times 10^{-5} & \text{-0.00029345} & \text{-0.0028605} \\ 7.9745 \times 10^{-5} & 0.0015719 & 0.012768 \\ \text{-0.00039017} & \text{-0.0064281} & \text{-0.043326} \\ 0.0015785 & 0.021449 & 0.11279 \\ \text{-0.0054324} & \text{-0.058119} & \text{-0.23438} \\ 0.016097 & 0.13210 & 0.38359 \\ \text{-0.041476} & \text{-0.24960} & \text{-0.47393} \\ 0.091780 & 0.39290 & 0.39510 \\ \text{-0.17778} & \text{-0.49152} & \text{-0.10852} \\ 0.29897 & 0.45559 & \text{-0.24409} \\ \text{-0.44078} & \text{-0.23769} & 0.37962 \\ 0.53922 & \text{-0.10088} & \text{-0.16311} \\ \text{-0.52197} & 0.36497 & \text{-0.22086} \\ 0.34568 & \text{-0.34533} & 0.33064 \\ \text{-0.10101} & 0.11494 & \text{-0.12827} \end{pmatrix}$$

Without application of Stage I, the matrix X_M did not satisfy the orthogonality constraint, as shown by the residual:

$$||H(X_M)||_F = 3.668 \times 10^7.$$

Application of Stage I yielded:

$$X_u = \begin{pmatrix} -1.6363 \times 10^{-5} & \text{-0.00029433} & \text{-0.0028606} \\ 9.6687 \times 10^{-5} & 0.0015758 & 0.012768 \\ \text{-0.00044692} & \text{-0.0064414} & \text{-0.043328} \\ 0.0017231 & 0.021483 & 0.11280 \\ \text{-0.0057238} & \text{-0.058189} & \text{-0.23441} \\ 0.016549 & 0.13220 & 0.38365 \\ \text{-0.041977} & \text{-0.24972} & \text{-0.47406} \\ 0.092072 & 0.39297 & 0.39535 \\ \text{-0.17760} & \text{-0.49144} & \text{-0.10893} \\ 0.29832 & 0.45532 & \text{-0.24353} \\ \text{-0.44009} & \text{-0.23728} & 0.37893 \\ 0.53906 & \text{-0.10129} & \text{-0.16238} \\ \text{-0.52253} & 0.36525 & \text{-0.22148} \\ 0.34638 & \text{-0.34546} & 0.33102 \\ \text{-0.10127} & 0.11497 & \text{-0.12838} \end{pmatrix}$$

The updated matrix X_u now satisfies the orthogonality constraint, as shown by the residual:

$$||H(X_u)||_F = 9.09143 \times 10^{-6}$$

Results of Stage II

With the updated eigenvector matrix X_u from Stage I:

- The updated matrix K_u was symmetric, as shown by the residual norm:
$$||K_u - K_u^T||_F = 9.897 \times 10^{-8}.$$
- The measured eigenvalues and the corrected measured eigenvectors were reproduced accurately by the updated model, as shown by the following residual:

$$||R(K_u)||_F = 5.795 \times 10^{-5}.$$

5. Conclusion

Currently an active research is underway on quadratic inverse eigenvalue problem (QIEP). There are still many unanswered questions. The major difficulties lie with constructing a quadratic pencil from the knowledge of partial eigenstructure and with a prescribed structure on symmetry, positive definiteness and sparsity. Two practical variations of such partially prescribed and structure preserving QIEP Aries in Active Vibration Control (AVC) and Finite Element Model updating (FEMU). These problems require that a given quadratic pencil with certain structure is modified with the help of a small number of eigenvalues and eigenvectors such that the modified pencil satisfies certain practical requirements. While AVC does not require that the original structure of the pencil is to be preserved in the modified pencil, such requirement is mandatory for FEMU, which makes the problem difficult to solve. Many methods for model updating have been proposed in the literature, but unfortunately, they are not capable of preserving the structure. Furthermore, these methods have been developed without investigating the solutions of the underlying inverse eigenvalue problems. As a result many of these existing techniques are not supported with sound mathematical justification. An attempt has been made in this paper to justify some of these techniques with mathematical theory, by solving the relevant inverse eigenvalue problems and develop new methods with these theories. However, development of structure preserving model updating methods, especially when the structure to be preserved is the sparsity of the original model, remains extremely difficult. Relevant mathematical theory for structure preserving partially prescribed quadratic inverse eigenvalue problem needs to be developed first.

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