INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 3, Number 1, Pages 115–124

NUMERICAL INVESTIGATION OF KRYLOV SUBSPACE METHODS FOR SOLVING NON-SYMMETRIC SYSTEMS OF LINEAR EQUATIONS WITH DOMINANT SKEW-SYMMETRIC PART

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(Communicated by Lubin Vulkov)

Abstract. Numerical investigation of BiCG and GMRES methods for solving non-symmetric linear equation systems with dominant skew-symmetric part has been presented. Numerical experiments were carried out for the linear system arising from a 5-point central difference approximation of the two dimensional convection-diffusion problem with different velocity coefficients and small parameter at the higher derivative. Behavior of BiCG and GMRES(10) has been compared for such kind of systems.

Key Words. convection-diffusion problem, central difference approximation, Krylov subspace methods, BiCG, GMRES(10), triangular preconditioners, non-symmetric systems, eigenvalue distribution of matrices

1. Introduction

The convection-diffusion equation is of much importance for modelling flow problems in computational fluid dynamics. While studying the property of a model convection-diffusion problem we can make some assumptions about the behavior of practical problems.

Let us consider the steady convection-diffusion problem:

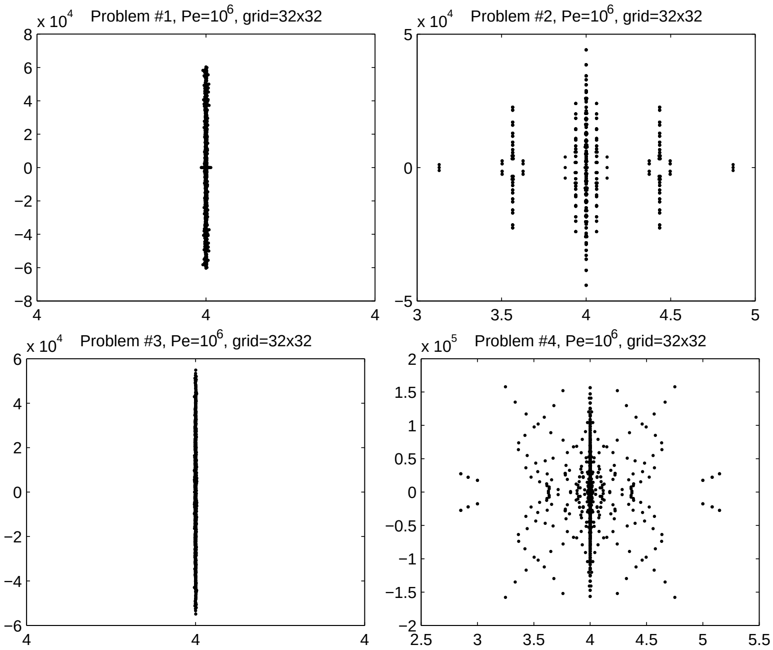
$$(1)\qquad \begin{cases} -Pe^{-1}\Delta u + \frac{1}{2}\{v_1 u_x + v_2 u_y + (v_1 u)_x + (v_2 u)_y\} = f, \\ u(x,y)|_{\partial\Omega} = 0, \quad div\,(\bar{v}) = 0, \quad \bar{v} = \{v_1, v_2\}, \\ (x,y) \in \Omega = [0,1] \times [0,1], \quad f = f(x,y), \quad u = u(x,y) \end{cases}$$

where Pe is Peclet number, $\bar{v} = \{v_1, v_2\}$ is velocity vector. The first term in (1) describes the diffusion process while other terms correspond to the convective process. The magnitude of dimensionless parameter Pe determines the ratio of the convection process to the diffusion one. When Pe is greater than a certain constant and boundary conditions are in disagreement with the right-hand side there arise singular perturbation problems with boundary and interior layers [2].

The choice of discretization method for problem (1) and appropriate iterative method for the corresponding linear system is very important. There are various ways to discretize (1). In the context of finite difference, the most widespread schemes are the central difference (second-order scheme) and the upwind (fires-order scheme). It is well known [2] that in general, linear system with M-matrix [12] can be obtained by applying the upwind schemes while positive real matrix

Received by the editors September 10, 2004 and, in revised form, May 11, 2005.
2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.
This research was supported by RFBR, grant N03-01-00005-a, and RU, grant N 03.01.272.

FIGURE 1. The eigenvalue distributions of the original matrix



can be obtained by using the central FD schemes [4] for first order derivatives. The upwind scheme yields M-matrix, and classical iterative methods converge in this case. We have used central difference approximation. In this case classical iterative methods for solution the resulted linear system may not converge when the Peclet number is greater than a certain constant.

To discretize the domain a mesh with meshsize h in both x and y direction was used.

When using natural ordering of unknowns, we have obtained system of linear equations with non-symmetric positive real matrix:

$$Au = f \tag{2}$$

where A is $(N-1) \times (N-1)$ matrix, u is the vector of unknown, f is the right-hand side.

In Figure 1 we depict the eigenvalue distribution of the matrix A obtained from approximation of equation (1) with various velocity coefficients (see Table 1) in order to compare it with the spectra of preconditioned matrices. We can see that the spectra of the matrices obtained from problems 1 and 3 have the same structure. All four spectrums are symmetric with respect to the point $(4, 0)$.

In this paper we present results of a preconditioned iterative solver based on BiCG for (2). First of all we compare GMRES(10) and BiCG. Further we compare the preconditioned BiCG with unpreconditioned BiCG. For completeness, we compared preconditioners proposed by us with popular SSOR precontitioner. The numerical tests were carried out on the grids 32×32, 64×64, 128×128 for all four problems (see Table 1). These test problems were borrowed from [1, 3]. The right-hand side function $f(x, y)$ was prescribed to satisfy the given exact solution $u(x, y) = e^{xy} \sin(\pi x) \sin(\pi y)$. Pe was altered between 10 and 10^6. According to the conventional classification [3] when $Pe \leq 10^3$ we get a moderately non-symmetric problem, otherwise when $Pe > 10^3$, we call it a strongly non-symmetric problem one (this boundary is vague). The initial guess was set to be a zero vector and the iterations were performed until $\|r^k\|_2/\|r^0\|_2 < 10^{-6}$, where r_k, r_0 are k-th and the initial residuals.

Table 1. Velocity coefficient for test problems

problem No	1	2	3	4
$v_1(x,y)$	1	$1-2x$	$x+y$	$\sin(\pi x)$
$v_2(x,y)$	-1	$2y-1$	$x-y$	$-\pi y\cos(\pi x)$

2. Krylov Subspace Methods

Recent methods for solving the discretized convection-diffusion equation are direct methods, generally the Gaussian elimination of some kind. However, for a large sparse linear systems iterative methods are more effective than the direct methods because iterative methods are usually the only means to find a solution with reasonable computational cost.

The best known iterative methods for solving partial differential equations are the relaxation-type methods. Typical examples are the Jacobi, Gauss-Seidel, and SOR methods. Their performance is highly dependent on the diagonal dominance of the coefficient matrices, the meshsize and the boundary conditions. For the SOR method, the estimate of the optimal over-relaxation parameter for general problems is still an open question. Besides many iterative methods are of convergence difficulty when they are used to solve discretized convection-diffusion equation with large Pe. For these and other reasons, recent focus on iterative methods has been shifted to favor the so-called parameter-free methods, such as Krylov subspace methods.

In this paper, we primarily resort to biconjugate gradient method (BiCG) [10, 11]. It belongs to Krylov subspace methods.

We solve linear systems with different Pe and h using BiCG and GMRES(10) [9]. To estimate the efficiency of the method the number of iterations has been used. (see Table 2). By the results obtained we can make the following conclusions.

Behavior of GMRES(10) and BiCG is quite different for various types of problems and is closely connected with the type of velocity coefficients. The most difficult problem for BiCG is problem 3 and for GMRES(10) is problem 4.

BiCG method solves system (2) well enough including the cases when the matrix loses diagonal dominance. However it has irregular convergence (see Figure 2).

The numerical experiments show that the number of iterations of unpreconditioned GMRES(10) depends on $R_h = Pe * h/2$. GMRES(10) works fast for problem 1 (constant velocity) and problem 2 (weakly changing velocity). It was established that when R_h is greater than a certain constant it's magnitude does not affect the BiCG convergence rate. On the other hand BiCG method is sensitive to grid size. The greater the grid size we use the more iterations are necessary for BiCG method to converge. The worst convergence BiCG method has for problem 3 while it works fast for problem 2.

As has been mentioned above that the number of iterations of BiCG method depends on the size of the linear system, while GMRES(10) method is not sensitive to it.

TABLE 2. The number of BiCG and GMRES(10) iterations

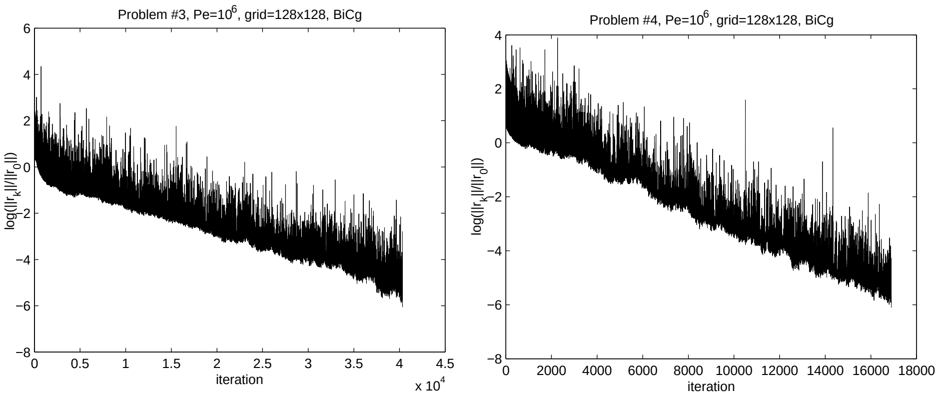
		32×32			64×64	
Pe	$R_h = Pe * h/2$	$BiCG$	$GMRES(10)$	$R_h = Pe * h/2$	$BiCG$	$GMRES(10)$
$Problem 1$						
10	0.151515	4	1	0.076923	3	1
10^3	15.1515	139	23	7.6923	97	10
10^5	1515.15	771	1469	769.23	831	934
10^7	151515	979	> 50000	76923	2307	45478
$Problem 2$						
10	0.151515	4	1	0.076923	3	1
10^3	15.1515	99	16	7.6923	69	8
10^5	1515.15	207	1133	769.23	831	855
10^7	151515	331	> 50000	76923	1019	40274
$Problem 3$						
10	0.151515	4	1	0.076923	3	1
10^3	15.1515	188	21	7.6923	96	10
10^5	1515.15	2492	1570	769.23	8001	992
10^7	151515	3468	> 50000	76923	15918	> 50000
$Problem 4$						
10	0.151515	7	1	0.076923	5	1
10^3	15.1515	470	59	7.6923	311	33
10^5	1515.15	1666	4433	769.23	6000	2845
10^7	151515	1732	> 50000	76923	6172	> 50000

3. Preconditioners for non-symmetric linear systems

In order to improve the convergence of iterative methods the matrix is transformed to another one by a suitable linear transformation. This process is known as preconditioning. Instead of (2) we solve the preconditioned linear system

$$B^{-1}Ax = B^{-1}b.$$

FIGURE 2. BiCG for problems 3 and 4



We have introduced left-side preconditioning. One may also use right-side preconditioning, i.e. formally solve

$$AB^{-1}z = b$$

and get x from $x = B^{-1}z$ or even use both right-side and left-side techniques. The paper presents left-side preconditioning.

TABLE 3. Triangular and product triangular preconditioners

$Method$	$operator B$	
$DTKM$	$B = B_C + 2K_U$	$B_C = \alpha_i E$
$PTKM$	$B = (B_C + 2K_L){B_C}^{-1}(B_C + 2K_U)$	$B_C = E$

It must be pointed out that the main requirements of a good preconditioner are the following [8]: the product $B^{-1}A$ must be close to the identity, B should be easily inverted (for instance, B is a diagonal or triangular matrix), the preconditioned system is solved easily and faster than the original system.

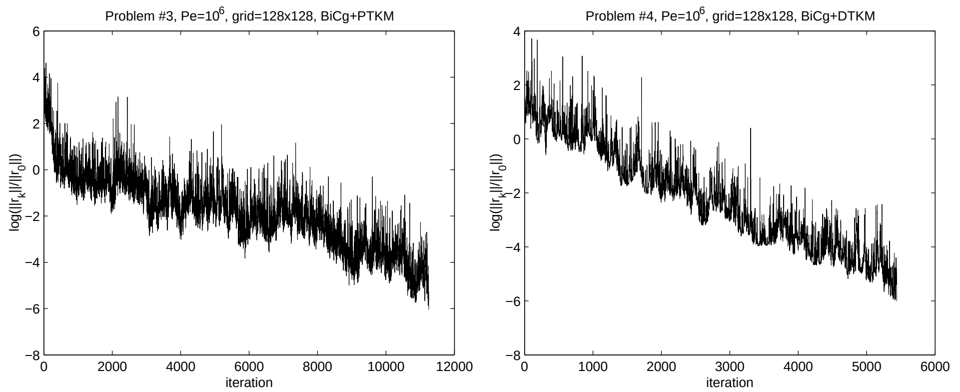
The basic idea of construction the triangular and product triangular preconditioners proposed by us was put forward in [5, 6, 7]. We have used the skew-symmetric part of the matrix A from system (2) (each non-symmetric matrix can be represented as a sum of symmetric and skew-symmetric part) and, it is required the matrix to be positive real.

Consider the ways to choose operator B (see Table 3). Here K_U is an upper and K_L is lower triangular part of skew-symmetric part of matrix A. B_C is a symmetric matrix which is constructed in a special way; the parameter $\alpha_i > 0$ are chosen in compliance with the formula:

$$\alpha_i = \sum_{j=1}^{n} |m_{ij}| \quad i = 0, ..., n, \quad \alpha_i E = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$$

here M is a symmetric matrix, which is constructed by the formula $M = A_0 + K_L - K_U$.

FIGURE 3. BiCG+preconditioner for problems 3 and 4



The numerical results for preconditioned BiCG are listed in Tables 4 and 5 for the optimal parameter $\tilde{\tau}$. It provides the number of preconditioned BiCG iterations

FIGURE 4. Eigenvalue distribution of $B^{-1}A$, $\quad B = DTKM(A)$

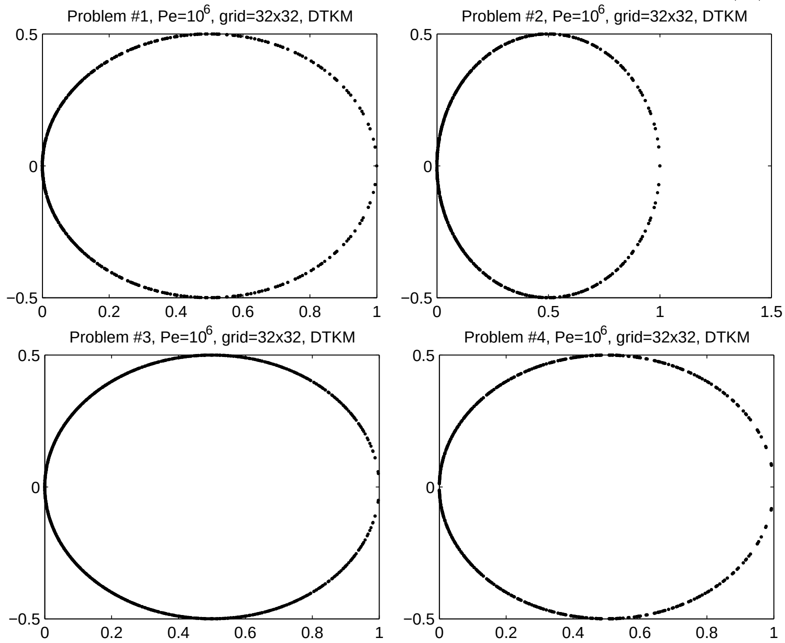


FIGURE 5. Eigenvalues distribution histogram of $B^{-1}A$, $\quad B = DTKM(A)$

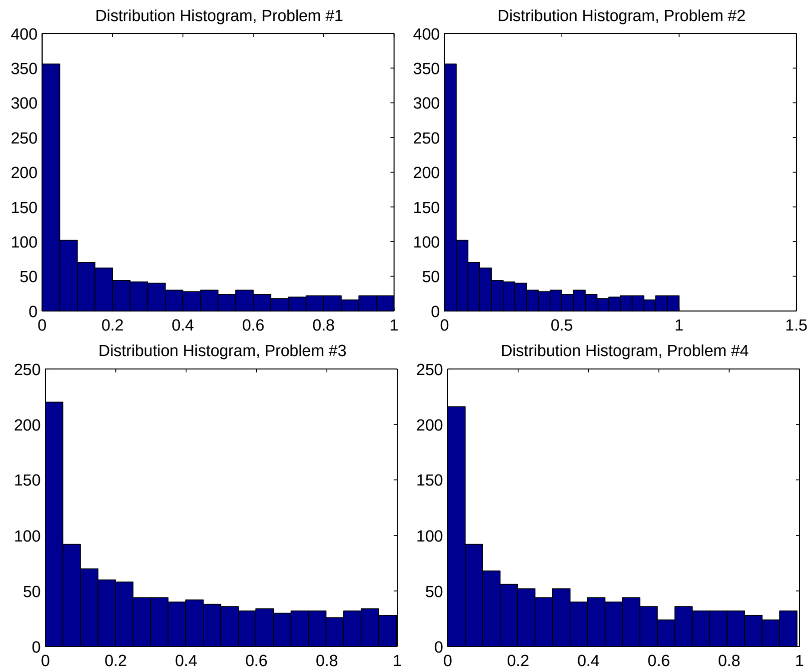


FIGURE 6. Eigenvalue distribution of $B^{-1}A$, $B = PTKM(A)$

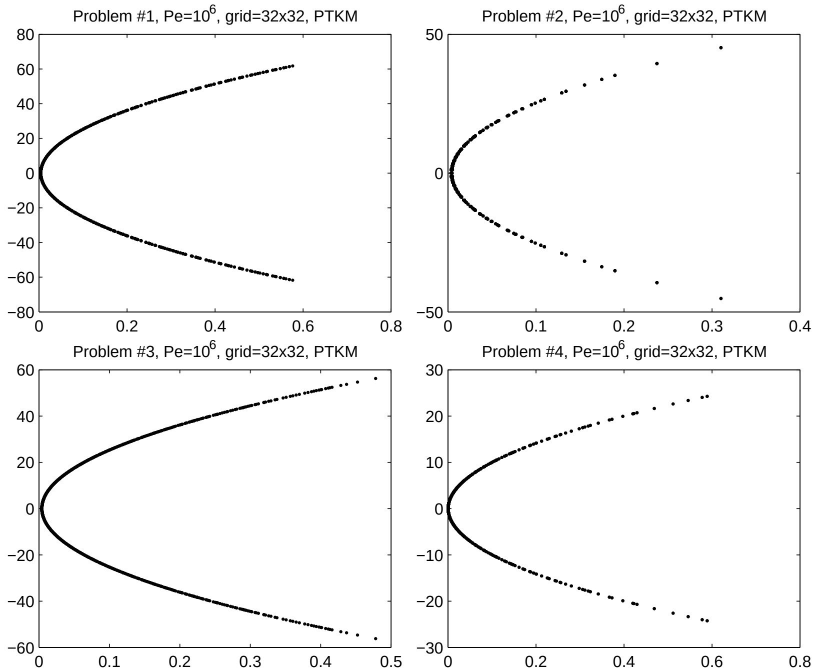
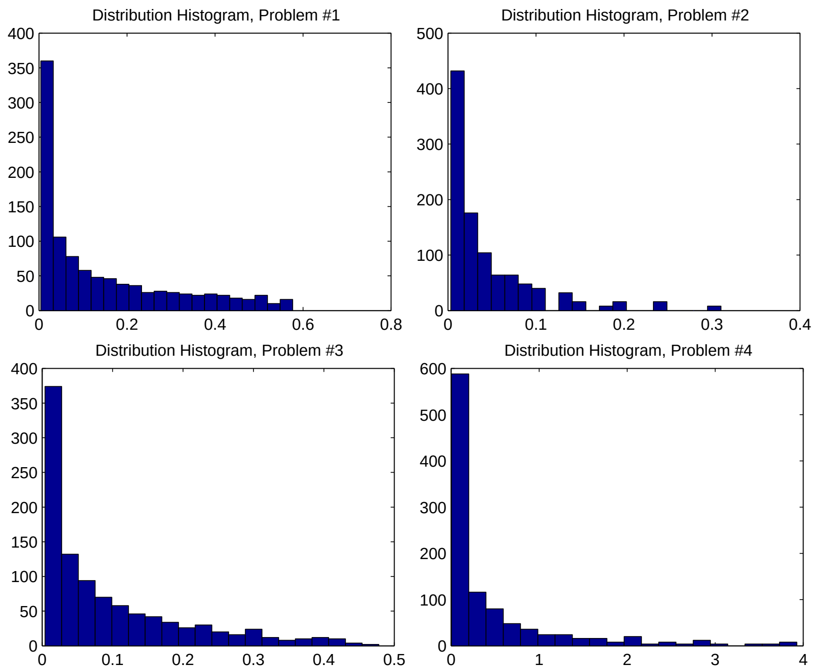


FIGURE 7. Eigenvalues distribution histogram of $B^{-1}A$, $B = PTKM(A)$



(N). According to these results the most effective preconditioner for BiCG method is PTKM for problem 3 and DTKM for problem 4 In Figure 3 we consider a behavior of the residual for this problems.

So, we can recommend to use BiCG without preconditioners for simple problems (problems 1 and 2) and with preconditioners (DTKM or PTKM) for hard ones (problems 3 and 4).

In Figure 4 and 6 we depict spectrum of $B^{-1}A$. We see that the spectrum of the preconditioned matrix is enclosed to an ellipse centered in $(0.5, 0)$, when $B = DTKM(A)$ (B is the DTKM preconditioner for matrix A). When $B = PTKM(A)$ (B is the PTKM preconditioner for matrix A) the spectrum is enclosed to a semi-ellipse centered in $(c, 0)$, where c is depends on the problem number. Also note the change of a scale and spectral radii of the preconditioned matrix. In figure 5 and 7 we can see that the eigenvalues of the preconditioned matrices $B^{-1}A$ are clustered at zero.

Table 4. The number of BiCG iterations with and without preconditioners

	64×64			
Pe	$BiCG$	$BiCG + DTKM$	$BiCG + PTKM$	$BiCG + SSOR$
$Problem1$				
10^2	50	9	5	6
10^3	470	96	40	38
10^4	1618	372	407	263
10^5	1666	1362	763	689
10^6	1702	1567	1329	1377
$Problem2$				
10^2	7	11	5	6
10^3	69	55	24	21
10^4	557	244	207	216
10^5	831	615	837	893
10^6	929	712	939	985
$Problem3$				
10^2	9	10	5	6
10^3	96	97	25	32
10^4	1052	579	260	238
10^5	8001	3748	2263	2401
10^6	11769	6076	6270	7608
$Problem4$				
10^2	26	32	9	10
10^3	311	148	69	72
10^4	2679	828	721	668
10^5	6000	2096	2546	2737
10^6	6054	2212	2983	3327

Table 5. The number of BiCG iterations with and without preconditioners

Pe	128×128			
	$BiCG$	$BiCG + DTKM$	$BiCG + PTKM$	$BiCG + SSOR$
		$Problem1$		
10^2	5	5	3	5
10^3	39	39	15	13
10^4	379	361	379	246
10^5	815	783	811	755
10^6	1287	1565	1283	1289
		$Problem2$		
10^2	4	8	4	4
10^3	34	37	12	12
10^4	439	439	228	161
10^5	2243	896	3560	2317
10^6	2751	1669	2717	2785
		$Problem3$		
10^2	5	7	4	5
10^3	43	42	11	15
10^4	562	319	178	181
10^5	5472	2354	1653	1623
10^6	40380	15889	11250	11818
		$Problem4$		
10^2	12	15	5	7
10^3	150	128	54	39
10^4	1779	529	721	505
10^5	11360	3098	5137	4144
10^6	16898	5439	11350	8213

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