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A solution of the affine quadratic inverse eigenvalue problem

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ABSTRACT

The quadratic inverse eigenvalue problem (QIEP) is to find the three matrices M, C, and K, given a set of numbers, closed under complex conjugations, such that these numbers become the eigenvalues of the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$. The affine inverse quadratic eigenvalue problem (AQIEP) is the QIEP with an additional constraint that the coefficient matrices belong to an affine family, that is, these matrices are linear combinations of substructured matrices. An affine family of matrices very often arise in vibration engineering modeling and analysis. Research on QIEP and AQIEP are still at developing stage. In this paper, we propose three methods and the associated mathematical theories for solving AQIEP: A Newton method, an alternating projections method, and a hybrid method combining the two. Validity of these methods are illustrated with results on numerical experiments on a spring-mass problem and comparisons are made with these three methods amongst themselves and with another Newton method developed by Elhay and Ram (2002) [12]. The results of our experiments show that the hybrid method takes much smaller number of iterations and converges faster than any of these methods.

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1. Introduction

The quadratic inverse eigenvalue problem (QIEP) is to find the matrices M, C and K such that the quadratic matrix pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ has a specified spectrum. QIEP arises in a wide variety of practical applications including control of mechanical vibrations, finite element model updating, acoustic studies, etc. [3–5,7,8–10,14,25].

Often, the matrices M, C and K are linear functions of the physical parameters [16], that is, they are given in the form:

$$M = M(\nu) = M_0 + \sum_{i=1}^{n} \nu_i M_i,$$

$$C = C(\alpha) = C_0 + \sum_{i=1}^{n} \alpha_i C_i,$$

$$C = K(\beta) = K_0 + \sum_{i=1}^{n} \beta_i K_i, \tag{1}$$

Such a matrix pencil is known as the *affine quadratic matrix pencil*. Since the matrix M is very often accurately estimated, from now on, we will assume that matrix M is constant, i.e. $M = M_0$. We will also assume that M is nonsingular.

Setting $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, the above pencil can be conveniently denoted by $(M, C(\alpha), K(\beta))$, and the eigenvalues of the pencil are similarly denoted by

$$\lambda(\alpha, \beta) = \{\lambda_1(\alpha, \beta), \ldots, \lambda_{2n}(\alpha, \beta)\}.$$

To see how such a pencil arises in applications, consider the following spring-mass system with damping (Fig. 1).

Here the n masses $m_1, \ldots, m_n$, are serially connected and the resistance to displacement is provided by springs with stiffness constants $\beta_1, \ldots, \beta_n$, respectively. The energy dissipation mechanism is represented by dampers with coefficients $\alpha_1, \ldots, \alpha_n$. Assuming that the damping is proportional to velocity $\dot{q}_i(t)$, the free vibrations (with no external forces) of the mass-spring system is governed by the system of second order differential equations of the form

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0, \tag{2}$$

with matrices M, C, and K given by

$$M = \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix},$$

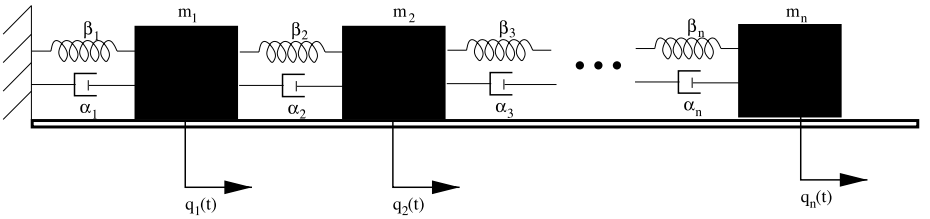

Fig. 1. Serially linked mass-spring system.

$$C = \begin{pmatrix} \alpha_1+\alpha_2 & -\alpha_2 & & \\ -\alpha_2 & \alpha_2+\alpha_3 & -\alpha_3 & \\ & \ddots & \ddots & \\ & & -\alpha_n & \alpha_n \end{pmatrix},$$

$$K = \begin{pmatrix} \beta_1+\beta_2 & -\beta_2 & & \\ -\beta_2 & \beta_2+\beta_3 & -\beta_3 & \\ & \ddots & \ddots & \\ & & -\beta_n & \beta_n \end{pmatrix}. \tag{3}$$

The **natural frequencies** are related to the eigenvalues of the associated quadratic matrix pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ and the eigenvectors are called the **mode shapes** or just the **modes**.

Note that the coefficient matrices M, C and K of the above system are structured: M is diagonal, C and K are tridiagonal. Moreover, all three of them are symmetric. The matrices C and K are clearly the members of the affine families of the form (1), with

$$K_0 = C_0 = 0, \quad K_1 = C_1 = e_1 e_1^T$$
$$K_i = C_i = (e_{i-1} - e_i)(e_{i-1} - e_i)^T, \quad i = 2, 3, \ldots, n.$$

In this paper we consider QIEP for such an affine pencil without imposing any structures on matrices C and K, except for the symmetry, assuming that such a solution exists [20,23,26]. Specifically, the affine quadratic inverse eigenvalue problem (AQIEP) to be considered in this paper is defined as follows.

Problem 1 (*Affine Quadratic Inverse Eigenvalue Problem (AQIEP)*)**.** Given a set with distinct entries $\{\mu_1, \ldots, \mu_{2n}\}$, closed under complex conjugation, find $(\alpha, \beta) \in \mathbb{R}^{2n}$, such that $\{\mu_1, \ldots, \mu_{2n}\}$ are eigenvalues of the pencil $(M, C(\alpha), K(\beta))$.

We will call the set $\{\mu_1, \ldots, \mu_{2n}\}$, the **target set of eigenvalues**. The matrix of target eigenvalues is denoted by Σ.

We first develop a Newton's method for solving Problem 1. The method consists of finding a zero (α, β) of the function:

$$f(\alpha, \beta) = \begin{pmatrix} \lambda_{\sigma(1)}(\alpha, \beta) - \mu_1 \\ \vdots \\ \lambda_{\sigma(2n)}(\alpha, \beta) - \mu_{2n} \end{pmatrix}, \tag{4}$$

where $\{\mu_1, \ldots, \mu_{2n}\}$ are the target eigenvalues and $\{\lambda_1(\alpha, \beta), \ldots, \lambda_{2n}(\alpha, \beta)\}$ are the eigenvalues of the pencil $(M, C(\alpha), K(\beta))$ in some order, and σ chosen in such a way that $\sum_i |\lambda_{\sigma(i)}(\alpha, \beta) - \mu_i|$ is minimum over all permutations σ of the set $\{1, 2, \ldots, 2n\}$.

In the well-known paper, Friedland et al. [15] considered the standard affine inverse eigenvalue problem for the matrix A of the form

$$A = A_0 + \sum_{i=1}^{n} \alpha_i A_i,$$

where $A_i = A_i^T \in \mathbb{R}^{n \times n}$. That is, the inverse problem considered there finds the scalars α_k, $k = 1, \ldots, n$, such that the matrix A has a prescribed set of n eigenvalues.

The authors described four Newton-type iterative methods and discussed their local convergence properties. In order to efficiently apply a Newton method to the function of type (4), it is important that the eigenvalues are reordered in a suitable way. The right ordering of the eigenvalues is a nontrivial task if the matrices A_i are non-symmetric, due to the fact that the eigenvalues in this case are complex in general. To avoid the ordering problem, Elhay and Ram [12] proposed to extend a Newton's method described in [15] to solve Problem 1 by finding a zero of the function:

$$g(\alpha) = \begin{pmatrix} f_1(\alpha) \\ \vdots \\ f_n(\alpha) \end{pmatrix} = \begin{pmatrix} \det(A_0 - \mu_1 I_n + \sum_{k=1}^{n} \alpha_k A_k) \\ \vdots \\ \det(A_0 - \mu_n I_n + \sum_{k=1}^{n} \alpha_k A_k) \end{pmatrix} \tag{5}$$

to higher-order polynomials.

Later in [13], Elhay and Ram proposed a Newton method for solving QIEP by finding a zero of the function (4), under the assumption that the number of real and complex eigenvalues in each iterate remains the same as the number of real and complex eigenvalues in the target set. No such assumptions are made in the present paper. Also, the ordering problem was not addressed in [13].

Newton's method, when applied to the function (4), requires evaluation of the Jacobian. In section 3, a technique, based on the orthogonality relations of the eigenvectors of the quadratic pencil, is proposed for this purpose.

Besides Newton's method, we also propose an *alternating projections method* for the AQIEP. An alternating projection method requires computation of the projection operators. In Section 4 we show how these projections can be computed using an optimization technique for the matrix nearness problem. Finally in Section 6, we propose a hybrid method by combining Newton's method with the alternating projections method. In this hybrid computation, the alternating projections method is used to generate a good initial approximation for the Newton method to ensure its local convergence.

Since the eigenvalues of the quadratic pencil $P(\lambda)$ can be complex, even when the matrices M, C and K are symmetric, an implementation of Newton's method, as said before, requires a reordering of the eigenvalues. The reordering problem is effectively a *matching problem* for a two sets of given numbers. This is discussed in Section 2. The matching problem can easily be solved when the eigenvalues are real (see Theorem 2.1). For reordering a set of complex eigenvalues, the so-called Hungarian method [19] can be used and this is the method we have used in our numerical experiments.

A comparison has also been made between the proposed Newton's method and the Elhay–Ram method developed in [12]. The results of Newton's method on our experiment on the spring-mass problem again compare favorably with those of the Elhay–Ram method.

2. Ordering the eigenvalues: matching problem

The problem of finding the permutation σ, needed to evaluate the function f in (4), is effectively a *matching problem* for two given sets of numbers, which is defined as follows: Given the two sets of numbers, $a = \{a_1, \ldots, a_k\}$ and $b = \{b_1, \ldots, b_k\}$, the set $\{a_{\sigma(1)}, \ldots, a_{\sigma(k)}\}$ matches the set $\{b_1, \ldots, b_k\}$ if σ is a permutation which minimizes

$$\sum_{j=1}^{k} |a_{\sigma(j)} - b_j|.$$

Thus, the two sets of eigenvalues $\{\lambda_1, \ldots, \lambda_{2n}\}$ and $\{\mu_1, \ldots, \mu_{2n}\}$ will match if

$$\sum_{j=1}^{2n} |\lambda_{\sigma(j)}(\alpha, \beta) - \mu_j| \tag{6}$$

is minimized among all possible permutations of the list of eigenvalues $\lambda_i(\alpha, \beta)$:

$$\sigma \in \arg \min_{\sigma(1),\ldots,\sigma(n!)} \sum_{j=1}^{2n} |\lambda_{\sigma(j)}(\alpha, \beta) - \mu_j|. \tag{7}$$

In other words, $\{\lambda_{\sigma_1}(\alpha, \beta), \ldots \lambda_{\sigma_{2n}}(\alpha, \beta)\}$ is the closest match for the target eigenvalues $\{\mu_1, \ldots, \mu_{2n}\}$.

In the context of ordering the real sets, we recall the well-known result [18].

Theorem 2.1 [18]. *Given two sets of real numbers* $a = \{a_1, \ldots, a_k\}$ *and* $b = \{b_1, \ldots, b_k\}$, *the expression*

$$\sum_{j=1}^{k} |a_j - b_j|$$

has the minimum value, when a and b are both monotonically increasing or both monotonically decreasing, i.e., $a_1 \leqslant \cdots \leqslant a_k$ *and* $b_1 \leqslant \cdots \leqslant b_k$ *or* $a_1 \geqslant \cdots \geqslant a_k$ *and* $b_1 \geqslant \cdots \geqslant b_k$.

For our numerical experiments, to solve the matching problem for two sets of complex numbers we have used the so-called Hungarian method [19]. This is an $O(n^3)$ method.

The Hungarian algorithm can be described as follows.

Algorithm 1. The Hungarian Method

INPUT: $a = (a_1, \ldots, a_n),\ b = (b_1, \ldots, b_n)$
OUTPUT: s = permutation vector which solves the assignment problem

1: Form the cost matrix C, $C_{ij} = |a_i - b_j|$
2: In each row choose a minimum and subtract that number from every row entry
3: In each column choose a minimum and subtract that number from every column entry
4: Choose the least number of columns and rows so that all of the zero entries are taken into account. Denote such a set of columns by I and set of rows by J, i.e. if $C_{ij} = 0$ then either $i \in I$ or $j \in J$.
5: If the number of chosen rows and columns is n, then a_i is assigned to b_j if $C_{i,j} = 0$, and the assignment is done. If the number of chosen rows and columns is less than n, go to the next step
6: Find

$$a = \min_{i \in I^C,\ j \in J^C} C_{ij},\ I^C = \{1, 2, \ldots, n\} - I,\ J^C = \{1, 2, \ldots, n\} - J.$$

Then update the cost matrix $C_{ij} = C_{ij} - a, i \in I^C,\ j \in J^C$ and $C_{ij} = C_{ij} + a, i \in I,\ j \in J$. Go to step 4.

3. Newton's method

In this section we show how the derivatives required for computation of the Jacobian of Newton's method can be computed by exploiting an orthogonality relation for the eigenvectors of the quadratic pencil $(M, C(\alpha), K(\beta))$. This orthogonality relation was originally proved in [6] and a real valued representation was obtained in [8]. See also [2]. First, we note that $f(\alpha, \beta)$ is a differentiable function in the neighborhood of a solution.

Theorem 3.1. *Let* (α^*, β^*) *be a solution to Problem* 1. *Then there exists a neighborhood of* (α^*, β^*) *which contains no singular points of the spectra. These are the points where the pencil has multiple eigenvalues*.

For details see [1,17,24].

Corollary 3.2. *There is a neighborhood of* (α^*, β^*) *where* $\lambda_i(\alpha, \beta)$ *are distinct and are differentiable functions*.

Thus, the function f defined above in (4) is a differentiable function in a neighborhood of a solution.

Let Λ and X be, respectively the matrices of eigenvalues and eigenvectors of the quadratic pencil (M, C, K).

To develop a computable formula for the Jacobian for the function $f(\alpha, \beta)$, let us first rewrite the quadratic eigenvalue problem:

$$M\Lambda(\alpha, \beta)^2 X(\alpha, \beta) + C(\alpha)\Lambda(\alpha, \beta)X(\alpha, \beta) + K(\beta)X(\alpha, \beta) = 0$$

as the symmetric generalized eigenvalue problem:

$$\begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} X \\ X\Lambda \end{pmatrix} = \begin{pmatrix} C & M \\ M & O \end{pmatrix} \begin{pmatrix} X \\ X\Lambda \end{pmatrix} \Lambda.$$

It is understood that the matrices K, C, Λ, and X are functions of α and β.

In [6], the following orthogonality relations for the matrix $col(X, X\Lambda)$ have been derived:

$$\begin{pmatrix} X \\ X\Lambda \end{pmatrix}^T \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} X \\ X\Lambda \end{pmatrix} = D$$

and

$$\begin{pmatrix} X \\ X\Lambda \end{pmatrix}^T \begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} X \\ X\Lambda \end{pmatrix} = D\Lambda,$$

where D is some diagonal matrix. Without any loss of generality, the matrix D can be scaled to the identity matrix. Let x_i denote the ith column of the matrix X. Then a diagonal element, the matrix D is of the form $d_{ii} = x_i^T(2\lambda_i M + C)x_i \neq 0, \ \ i = 1, \ldots, 2n$. Setting now

$$z_i = \frac{x_i}{\sqrt{x_i^T(2\lambda_i M + C)x_i}},$$

the above orthogonality relations become:

$$\begin{aligned}
&z_i^T(2\lambda_i M + C)z_i = 1, \\
&z_i^T(\lambda_i^2 M - K)z_i = \lambda_i, \\
&\dot{z}_i^T(2\lambda_i M + C)z_i + z_i^T(2\dot{\lambda_i} M + \dot{C})z_i + z_i^T(2\lambda_i M + C)\dot{z}_i = 0, \qquad (8) \\
&\dot{z}_i^T(\lambda_i^2 M - K)z_i + z_i^T(2\dot{\lambda_i}\lambda M - \dot{K})z_i + z_i^T(\lambda_i^2 M - K)\dot{z}_i = 0. \qquad (9)
\end{aligned}$$

Here, $\dot{z}$ denotes the derivative of z with respect to α_i or β_i. Multiplying (8) by λ and subtracting it from (9) we obtain

$$\dot{\lambda_i} = -z_i^T(\lambda\dot{C} + \dot{K})z_i. \qquad (10)$$

Since C does not depend on β, and K does not depend on α, we have the following partial derivatives:

$$\frac{\partial\lambda_i}{\partial\alpha_k} = -\lambda_i z_i^T C_k z_i, \quad \frac{\partial\lambda_i}{\partial\beta_k} = -z_i^T K_k z_i.$$

These partial derivatives can be easily modified in case the matrix D is not an identity matrix. Substituting the value of z_i from above, we have

$$\frac{\partial\lambda_i}{\partial\alpha_k} = -\frac{\lambda_i x_i^T C_k x_i}{x_i^T(2\lambda_i M + C)x_i}, \quad \frac{\partial\lambda_i}{\partial\beta_k} = -\frac{x_i^T K_k x_i}{x_i^T(2\lambda_i M + C)x_i}.$$

Thus, the Jacobian $J = (J_{i,k})_{i,k=1}^n$ of f is given by

$$J_{ik} = \begin{cases} \frac{\partial\lambda_i}{\partial\alpha_k} = -\frac{\lambda_i x_i^T C_k x_i}{x_i^T(2\lambda_i M + C)x_i}, & k = 1, \ldots, n, \\ \frac{\partial\lambda_i}{\partial\beta_{k-n}} = -\frac{x_i^T K_{k-n} x_i}{x_i^T(2\lambda_i M + C)x_i}, & k = n+1, \ldots, 2n, \end{cases} \qquad (11)$$

and ith step of Newton's method can be defined as follows:

$$J(\alpha^i, \beta^i)\begin{pmatrix} \alpha^{i+1} - \alpha^i \\ \beta^{i+1} - \beta^i \end{pmatrix} = -f(\alpha^i, \beta^i). \qquad (12)$$

Algorithm 2. Newton's Method

INPUT: (i) λ^* — The target set of eigenvalues
(ii) $(\alpha^0,\ \beta^0)$ — An initial approximation
(iii) ϵ — A Tolerance
OUTPUT: $(\alpha^*,\ \beta^*)$ — An approximate solution to (4).

1: **for** i=0,1,... until convergence **do**
2: Find eigenvalues and eigenvectors of $(M, C(\alpha^i), K(\beta^i))$.
3: Solve the combinatorial minimization problem (6) and compute $f(\alpha^i, \beta^i)$
4: Calculate $J(\alpha^i, \beta^i)$ as given in (11) and compute $(\alpha^{i+1}, \beta^{i+1})$ by solving (12)
5: Stop if $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon$
6: **end for**

Remarks 1. (i) As stated before, the matching problem in Step 3 may be solved using so-called Hungarian method [19]. Clearly, Algorithm 2 is computationally feasible if only a small number of iterations are needed for convergence. (ii) A method to compute the Jacobian of the type described above, which does not rely on the orthogonality relations was also reported in the paper by Elhay and Ram [13].

4. The matrix nearness problem

In this section, we show how to compute the projections needed at each iteration of the alternating projections method (to be described in the next section) by solving the following matrix nearness problem in an optimization setting.

We need to compute a projection of the pair $(C, K) \in \mathbb{R}^{2n^2}$ onto the following non-convex set:

$$\mathcal{L} = \{(C, K) \in \mathbb{R}^{2n^2} \mid MX\Sigma^2 + CX\Sigma + KX = 0 \mid \text{for some fixed M, the diagonal matrix } \Sigma, \text{and the matrix } X \text{ s.t } ||x_i|| = 1\}.$$

The projection onto $\mathcal{L}$ is a solution to the following problem:

Given the symmetric matrix pencil (M, C, K) and the matrix $\Sigma = diag(\mu_1, \ldots, \mu_{2n})$, find the matrices $\widehat{C}$ and $\widehat{K}$ such that

$$||C - \widehat{C}||^2 + ||K - \widehat{K}||^2 \to \min \tag{13}$$

$$\text{s.t. } MX\Sigma^2 + \widehat{C}\Sigma + \widehat{K}X = 0, \tag{14}$$

$$\widehat{C}, \widehat{K} \in R^{n\times n}, \quad X \in C^{n\times 2n},\ ||x_i|| = 1, \tag{15}$$

where x_i is the ith column of matrix X.

The constraint (14) can be eliminated by expressing $\widehat{C}$ and $\widehat{K}$ in terms of the matrices Σ and X using the following result derived in [17].

Given a nonsingular matrix M, a diagonal matrix Σ and some matrix X, such that

$$col(X, X\Sigma) = \begin{pmatrix} X \\ X\Sigma \end{pmatrix}$$

is a nonsingular matrix, then relation

$$MX\Sigma^2 + \widehat{C}X\Sigma + \widehat{K}X = 0$$

holds if and only if

$$\widehat{C} = -MX\Sigma^2 V_2, \tag{16}$$

$$\widehat{K} = -MX\Sigma^2 V_1, \tag{17}$$

where the matrices V_1 and V_2 are given by

$$V = \begin{pmatrix} X \\ X\Sigma \end{pmatrix}^{-1} = (V_1, V_2), \quad V_1, V_2 \in \mathbb{R}^{2n\times n}. \tag{18}$$

In view of (16) and (17), the above minimization problem (13) now becomes:

$$F(X) = ||C - MX\Sigma^2 V_2||^2 + ||K - MX\Sigma^2 V_1||^2 \to \min. \tag{19}$$

$$\text{s.t. } x_i = \bar{x}_j \text{ whenever } \mu_i = \bar{\mu}_j. \tag{20}$$

The constraint (20) guarantees that the matrices $\widehat{C}$ and $\widehat{K}$ given by (16) and (17) are real. Note, that function F is a real-valued function of complex variables, thus this function does not satisfy the Cauchy–Riemann relationship and is not a differentiable function of a complex variable. To overcome this difficulty, consider the real-valued representation of the matrices X and Σ. Let $X = (u_1 + iv_1, u_1 - iv_1, \ldots, u_l + iv_l, u_l - iv_l, x_{2l+1}, \ldots, x_{2n})$ and $\Sigma = diag(\delta_1 + i\gamma_1, \delta_1 - i\gamma_1, \ldots, \delta_l + i\gamma_l, \delta_l - i\gamma_l, \lambda_{2l+1}, \ldots, \lambda_k)$. Define $X_r = XS^{-1}$ and $\Sigma_r = S\Sigma S^{-1}$, where

$$S = diag(S_1, \ldots, S_l, S_{2l+1}, \ldots, S_{2n}), \quad \text{and,}$$
$$S_j = \begin{cases} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, & j = 1, \ldots, l, \\ 1, & j = 2l+1, \ldots, 2n. \end{cases} \tag{21}$$

Then

$$X_r = [u_1, v_1, \ldots, u_l, v_l, x_{2l+1}, \ldots, x_{2n}],$$

and

$$\Sigma_r = diag(\Sigma_1, \ldots, \Sigma_l, \Sigma_{2l+1}, \ldots, \Sigma_{2n}), \text{ with} \tag{22}$$
$$\Sigma_j = \begin{cases} \begin{pmatrix} \delta_i & \gamma_i \\ -\gamma_i & \delta_i \end{pmatrix}, & j = 1, \ldots, l, \\ \lambda_j, & j = 2l+1, \ldots, 2n. \end{cases}$$

In view (21) and (22), $F(X)$ now can be expressed as a real-valued function of a real variable. Since M is nonsingular, we define

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix}$$

as the block-companion matrix of the pencil $P(\lambda) = \lambda^2 M + \lambda C + K$. This lets us reformulate the optimization problem (19) and (20) as:

$$\min_{X_r} ||A - Z\Sigma_r Z^{-1}||^2 \tag{23}$$

$$\text{where } \quad Z = col(X_r; X_r\Sigma_r), \quad X_r \in \mathbb{R}^{n\times n}. \tag{24}$$

Consider now Problem (23) disregarding the form of the matrix Z in (24). Let $H = Z\Sigma_r Z^{-1}$; then

$$\begin{aligned} F(Z) &= ||A - Z\Sigma_r Z^{-1}||^2 = \langle A - H, A - H\rangle \\ &= \langle A, A\rangle + \langle H, H\rangle - 2\langle A, H\rangle = ||A||^2 + tr(H^T H) - 2tr(A^T H). \end{aligned}$$

The derivative of the function F then can be calculated as follows:

$$\frac{\partial}{\partial Z_{ij}} tr(A^T H) = tr\left(A^T \frac{\partial}{\partial Z_{ij}}\left(Z\Sigma_r Z^{-1}\right)\right) = tr\left(A^T\left(\frac{\partial Z}{\partial Z_{ij}}\Sigma_r Z^{-1} - Z\Sigma_r Z^{-1}\frac{\partial Z}{\partial Z_{ij}}Z^{-1}\right)\right).$$

Now

$$\nabla_Z tr(A^T H) = AZ^{-T}\Sigma_r - Z^{-T}\Sigma_r Z^T A Z^{-T},$$

and

$$\frac{\partial}{\partial Z_{ij}} tr(A^T H) = \frac{\partial}{\partial Z_{ij}} tr(H^T A) = tr\left(A^T \frac{\partial}{\partial Z_{ij}} H\right) = tr\left(\frac{\partial}{\partial Z_{ij}} H^T A\right).$$

Thus,

$$\frac{\partial}{\partial Z_{ij}} tr(H^T H) = tr\left(\frac{\partial}{\partial Z_{ij}} H^T H\right) + tr\left(H^T \frac{\partial}{\partial Z_{ij}} H\right) = 2(HZ^{-T}\Sigma_r - Z^{-T}\Sigma_r Z^T H Z^{-T}).$$

Therefore, the matrix of derivatives with respect to $Z_{i,j}$'s is:

$$\begin{aligned} F_Z &= 2(HZ^{-T}\Sigma_r - Z^{-T}\Sigma_r Z^T H Z^{-T}) \\ &\quad -2(AZ^{-T}\Sigma_r - Z^{-T}\Sigma_r Z^T A Z^{-T}) = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}. \end{aligned} \tag{25}$$

and

$$(F_Z)_{i,j} = \frac{\partial}{\partial Z_{ij}} F(Z).$$

Remarks 2. If A were a normal matrix we could have reduced the search space to the set of orthogonal matrices, i.e., we could have considered the solution of the form $Z\Sigma_r Z^T$, where $ZZ^T = I$. In this case the gradient of F is the tangent vector given by

$$\nabla_Z F = F_Z - ZF_Z^T Z = AZ\Sigma_r - Z\Sigma_r Z^T A Z.$$

See [11].

Thus, when Z is orthogonal, we have

$$\nabla_Z F = 0 \iff AZ\Sigma_r - Z\Sigma Z^T A Z = 0 \iff AH - HA = 0.$$

Since A and X have common eigenvectors, this leads us to the following result. If matrix A is normal, then the solution to:

$$\begin{aligned} &\min_Z ||A - Z\Sigma Z^T||^2 \\ &ZZ^T = I \end{aligned}$$

is given by $Z = Y$, where Y is the eigenvector matrix of A.

Note the minimum value of the function F in this case is

$$F(Y) = ||Y(\Lambda_r - \Sigma_r)Y^T||^2 = \sum(\lambda_i - \mu_i)^2.$$

The formula for the derivative (25) has been obtained disregarding the fact that $Z = col(X_r; X_r\Sigma_r)$. Taking this fact into consideration, the formula of the derivative is given by

$$F_{X_r} = J_1.$$

Having the analytical expression for the derivative, we can now solve the optimization Problems (23) and (24) using a gradient-based optimization algorithm, for example, the Broyden–Fletcher–Goldfarb–Shannon method (**BFGS**) [21]. Once the solution matrix X has been found, the solution to the original problem (19), $\widehat{C}$ and $\widehat{K}$, can be found by substituting X into (16) and (17). We should note, that the computational cost of solving the minimization problem (23) and (24) is high. However, our numerical experiments show that a good approximation to the solution to the nearness problem (19) can be obtained by choosing X to be equal to a permuted eigenvector matrix of the pencil (M, C, K). The solution, in this case, can be approximated by

$$\widehat{C} \approx -MY_\sigma \Sigma^2 U_2, \tag{26}$$

$$\widehat{K} \approx -MY_\sigma \Sigma^2 U_1, \tag{27}$$

where Y_σ is the permuted eigenvector matrix of the pencil (M, C, K) and the permutation σ is the same as defined in (7). The matrices U_1 and U_2 are given by

$$U = \begin{pmatrix} Y_\sigma \\ Y_\sigma \Sigma \end{pmatrix}^{-1} = (U_1, U_2),\ U_1, U_2 \in \mathbb{C}^{2n\times n}.$$

5. An alternating projections method

Now we turn our attention to the description of the **alternating projections method**. First, we reformulate Problem 1 in the following form (Problem 2) for this purpose.

Consider the two sets:

$$\mathcal{L} = \{(C, K) \in \mathbb{R}^{2n^2} \mid MX\Sigma^2 + CX\Sigma + KX = 0, \text{ for some fixed M and matrix } X \text{ s.t } ||x_i|| = 1\},$$

and

$$\mathcal{A} = \left\{ (C, K) \in \mathbb{R}^{2n^2} \mid C = C(\alpha) = C_0 + \sum_{i=1}^{n} \alpha_i C_i,\ K = K(\alpha) = K_0 + \sum_{i=1}^{n} \beta_i K_i \right\}.$$

Here M,C_i's and K_i's are given and fixed.

In view of these definitions, Problem 1 can now be reformulated as follows.

Problem 2. Find $(\alpha, \beta) \in \mathbb{R}^{2n}$ such that $(M, C(\alpha), K(\beta)) \in \mathcal{L} \cap \mathcal{A}$.

The following results form a basis of the alternating projections method to solve Problem 2 [27]:

Theorem 5.1. *Let B_1, B_2 be the two closed convex sets in a finite dimensional Hilbert space H, $B_1 \bigcap B_2 \neq \emptyset$ and let P_{B_1} and P_{B_2} denote projection operators onto B_1 and B_2, respectively. Then,*

$$\lim_{n\to\infty} (P_{B_1} P_{B_2})^n = \lim_{n\to\infty} (P_{B_2} P_{B_1})^n = P_{B_1 \cap B_2}.$$

Note, both of the sets $\mathcal{L}$, $\mathcal{A}$ are closed, and furthermore the set $\mathcal{L}$ is nonconvex. Thus, the alternating projections might not converge. However, the alternating projections never increases the distance between two successive iterates, as the following result shows.

Theorem 5.2. *Let B_1, B_2 be the two closed sets in a finitely dimensional Hilbert space H, $B_1 \bigcap B_1 \neq \emptyset$ and let $y \in B_2$. Let P_{B_1} and P_{B_2} be the same as defined in Theorem 5.1. If*

$$u_1 = P_{B_1}(y), \quad y_1 = P_{B_2}(u_1), \quad u_2 = P_{B_1}(y_1),$$

then

$$||u_2 - y_1|| \leqslant ||u_1 - y_1|| \leqslant ||u_1 - y||.$$

Corollary 5.3. *For any given $x_0 \in H$, $\{(P_{B_1} P_{B_2})^n (x_0)\}_{n=0}^{\infty}$ is a nondecreasing sequence.*

As we noted above, the accumulation point of $\{(P_{B_1} P_{B_2})^n (x_0)\}_{n=0}^{\infty}$ is not necessarily a solution to Problem 2. However, practical experience has shown that even if the alternating projections do not converge, an accumulation point is still close to a solution.

Next, we derive projection operators $P_{\mathcal{L}}$ and $P_{\mathcal{A}}$. The inner product in R^{2n^2} is defined in the standard way:

$$\langle (C_1, K_1), (C_2, K_2) \rangle = tr(C_1^T C_2 + K_1^T K_2).$$

Let $P_{\mathcal{A}}(C, K) = (C(\alpha), K(\beta))$, then the coefficients (α, β) are found as solutions of the following linear systems:

$$A_1\alpha = b_1, \quad A_2\beta = b_2, \tag{28}$$

where $(A_1)_{ij} = tr(C_i^T C_j)$, $(A_2)_{ij} = tr(K_i^T K_j)$ and $(b_1)_i = tr((C - C_0)^T C_i)$, $(b_2)_i = tr((K - K_0)^T K_i)$. Define

$$c : \mathcal{A} \to R^{2n}, \quad c(C, K) = (\alpha, \beta). \tag{29}$$

The projection onto $\mathcal{L}$ is the solution to the problem (13).

Note, that this definition of the projection operator onto $\mathcal{L}$ is different from the one usually used in numerical analysis and numerical linear algebra literature. The value of the operator can be found using a gradient search optimization routine or as an approximation to the solution given by (26) and (27). Thus,

$$P_{\mathcal{L}}(C, K) = (\widehat{C}, \widehat{K}) \approx (MX_\sigma \Sigma^2 \overline{V}_2, MX_\sigma \Sigma^2 \overline{V}_1), \tag{30}$$

where X_σ is the permuted eigenvector matrix of the pencil (M, C, K). The permutation σ is as defined in (7) and V_1 and V_2 are defined by:

$$V = \begin{pmatrix} X_\sigma \\ X_\sigma \Sigma \end{pmatrix}^{-1} = V_1, V_2), \quad V_1, V_2 \in \mathbb{R}^{2n \times n}. \tag{31}$$

Remarks 3. Our numerical experiments indicate that the convergence of the alternating projections method does not suffer when the approximations of the projected matrices $\widehat{C}$ and $\widehat{K}$ defined by (26), (27) are used. The above discussion lead to Algorithm 3, stated below.

Algorithm 3. An Alternating Projections Method for Problem 2

INPUT: (i) λ^* — The target set of eigenvalues
(ii) (α^0, β^0) — An Approximal Solution
(iii) ϵ — A tolerance
OUTPUT: $(\hat{\alpha}, \hat{\beta})$ - A Solution to Problem 2.

1: **for** i = 0,1,... **do**
2: Form the matrix $(C(\alpha^i), K(\beta^i))$
3: Compute eigenvalues and eigenvectors of $(M, C(\alpha^i), K(\beta^i))$ and compute σ by (7)
4: Form the matrix X_σ
5: Compute $(\hat{C}, \hat{K})$ by either (a) approximating them using (30) or (b) by solving the optimization problem defined by (13) - (15) with a gradient-based optimization algorithm, such as the BFGS, using the gradient formula given by (25).
6: Compute $(\alpha^{i+1}, \beta^{i+1})$, by solving (28)
7: Stop if $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon$
8: **end for**

Next, we state a reformulated version of Algorithm 3 by computing the eigenvectors with the well-known inverse iteration method [5].

Suppose that (α^i, β^i) is our current estimate, $X_\sigma^{(i)}$ is an approximation to $X_\sigma(\alpha^i, \beta^i)$, and the matrix of eigenvectors of $(M, C(\alpha^i), K(\beta^i))$, is arranged in the order defined by (7). Let $x^i_{\sigma(j)}$ be the jth column of $X_\sigma^{(i)}$. To find $(\alpha^{i+1}, \beta^{i+1})$, we first compute $(\widehat{C}, \widehat{K})$ as:

$$(\widehat{C}, \widehat{K}) = (-MX_\sigma^{(i)} \Sigma^2 V_2, -MX_\sigma^{(i)} \Sigma^2 V_1), \tag{32}$$

where

$$V = \begin{pmatrix} X_\sigma^{(i)} \\ X_\sigma^{(i)} \Sigma \end{pmatrix}^{-1},$$

and then calculate $(\alpha^{i+1}, \beta^{i+1})$ by solving (28). To update our approximation to the eigenvectors, we apply one step of inverse iteration: compute $u_j, j = 1, \ldots, 2n$ by:

$$\begin{pmatrix} 0 & I \\ K & C \end{pmatrix} \begin{pmatrix} u_j \\ z_j \end{pmatrix} = \mu_j \begin{pmatrix} I & 0 \\ 0 & -M \end{pmatrix} \begin{pmatrix} x^i_{\sigma(j)} \\ \mu_j x^i_{\sigma(j)} \end{pmatrix}, \quad i = 1, \ldots, 2n \tag{33}$$

and then define

$$x^{i+1}_{\sigma(j)} = \frac{u^j}{||u^j||}.$$

The elements $x^{i+1}_{\sigma(j)}$ determine the new matrix X^{i+1}_σ. The vector u_j can be obtained as a solution of the following $n \times n$ linear system (see [22]):

$$(\mu_j^2 M + \mu_j C + K)u_j = (C + 2\mu_j M)x^i_{\sigma(j)}. \tag{34}$$

This leads us to the following reformulated alternating projection method.

Algorithm 4. Reformulated Alternating Projections Method for Problem 2

INPUT: (i) λ^* — The target set of eigenvalues
(ii) (α^0, β^0) — An initial approximation
(iii) ϵ — A tolerance
OUTPUT: $(\hat{\alpha}, \hat{\beta})$ - An Approximate Solution to Problem 2.

1: Form $(C(\alpha^0), K(\beta^0))$.
2: Compute eigenvalues and eigenvectors of $(M, C(\alpha^0), K(\beta^0))$ and compute σ defined by (7)
3: Form the matrix $X^{(0)}_\sigma = X_\sigma(\alpha^0, \beta^0)$.
4: **for** i = 0,1,... **do**
5: Compute the eigenvalues and eigenvectors of $(M, C(\alpha^i), K(\beta^i))$ and compute σ defined by (7)
6: Form the matrix X_σ
7: Compute $(\hat{C}, \hat{K})$ by either (a) approximating it using (30) or (b) by applying a gradient based optimization algorithm using the gradient formula (25) to solve the optimization problem defined by (13) - (15).
8: Compute $(\alpha^{i+1}, \beta^{i+1})$, by solving (28).
9: Stop if $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon$.
10: Form $(C, K) = (C(\alpha^{i+1}), K(\beta^{i+1}))$.
11: Solve $2n$ linear systems:

$$((\mu_j)^2 M + \mu_j C + K)u_j = (C + 2\mu_j M)x^i_{\sigma(j)},\ j = 1, \ldots, 2n$$

and compute

$$x^{i+1}_{\sigma(j)} = \frac{u^j}{||u^j||}.$$

12: **end for**

6. The hybrid method

In this section, we state a hybrid method combining the reformulated alternating projections method (Algorithm 4) with the proposed Newton method (Algorithm 2). Algorithm 4 is first used to generate a reasonably good approximation to the problem, which is then used in Newton's method as an initial approximation to speed up the convergence.

Algorithm 5. The Hybrid Method for AQIEP

INPUT: (i) λ^* — The target set of eigenvalues
(ii) $(\alpha^0,\ \beta^0)$ — An initial approximation
(iii) ϵ_1, ϵ_2 — Tolerances for the Newton and the alternating projection methods, respectively
OUTPUT: (α^*, β^*), An Approximate solution to Problem 2.

for i=0,1,.... **do**
  Compute $(\alpha^{i+1}, \beta^{i+1}) = c((P_{\mathcal{A}}P_{\mathcal{L}})(C(\alpha^i), K(\beta^i)))$.
  stop if $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon_2$.
end for
set $(\alpha^0, \beta^0) = (\alpha^{i+1}, \beta^{i+1})$.
for i=0,1,... **do**
  Compute the Jacobian $J(\alpha^i, \beta^i)$ by (11) and then compute $(\alpha^{i+1}, \beta^{i+1})$ by (12).
  Stop if $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon_1$
end for

7. Numerical experiments

In this final section, we present the results of comparison of the following methods:

- The alternating projections method.
- The proposed Newton's method.
- The Elhay–Ram method [12].
- The hybrid method (Algorithm 5).

7.1. Results of comparison of the alternating projections and the hybrid methods

Consider the mass-spring system as depicted in Fig. 1 having three degrees for freedom with the following target set of eigenvalues:

$$\lambda^* = \{-0.0271 \pm i1.0108, -0.0177 \pm i0.6724, -0.0023 \pm i0.2658\}.$$

Let $\alpha^0 = (1, 1, 1),\ \beta^0 = (1, 1, 1)$. Then after applying three iterations of Algorithm 4, we obtain the following approximation of (α, β), denoted by α^{AP} and β^{AP}:

$$\alpha^{AP} = (0.0332, 0.0134, 0.0169), \quad \beta^{AP} = (0.7188, 0.2193, 0.1915).$$

The eigenvalues of the resulting affine pencil are given by

$$\lambda(\alpha^{AP}, \beta^{AP}) = \{-0.0271 \pm i1.0110, -0.0176 \pm i0.6724, -0.0021 \pm i0.2554\}.$$

Clearly, the above approximations are not accurate enough. Indeed, *as can be seen from Fig. 2, after a few iterations, the alternating projections method stagnates without any further improvement.*

Next, we apply Newton's method (Algorithm 2) with $(\alpha^{AP},\ \beta^{AP})$ as an initial approximation. After seven iterations, we obtain the following much improved value:

$$\alpha^N = (0.0139, 0.0203, 0.0199), \quad \beta^N = (0.6038, 0.2722, 0.1988).$$

The accuracy of the above approximation is easily verified by computing the norm of the difference between the target eigenvalue set and the eigenvalues obtained using the approximation (λ^N, β^N):

$$||\lambda^* - \lambda(\alpha^N, \beta^N)|| = 1.29 \times 10^{-9}.$$

We now compare the hybrid method with our proposed Newton method.

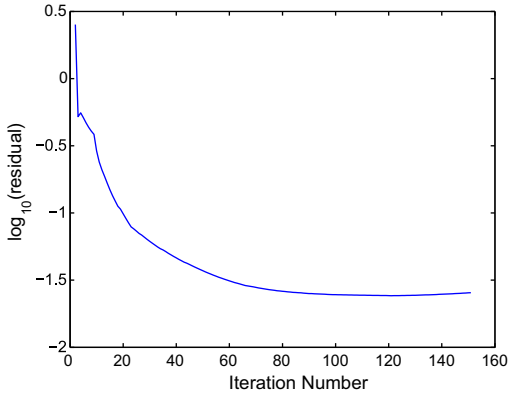

Fig. 2. Convergence of the alternating projections method (Algorithm 4).

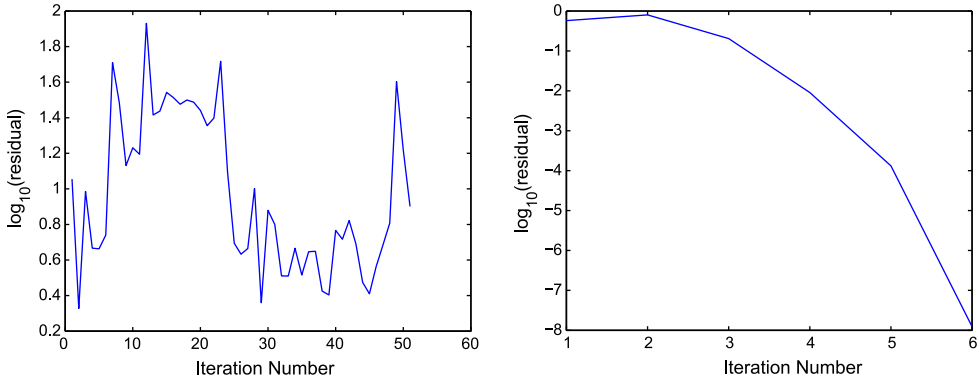

Fig. 3. Comparison of convergence of Newton's method (left) with the hybrid method (right).

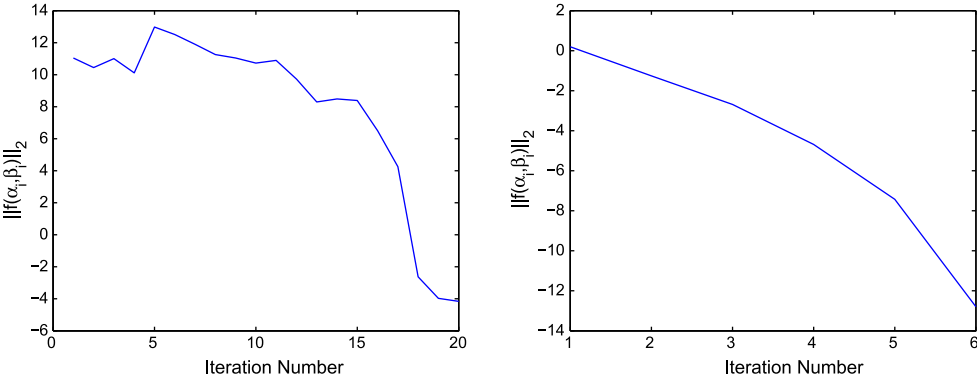

Fig. 4. Comparison of convergence of the proposed Newton's method (right) with the Elhay–Ram method (left) [12].

This time, we use the same spring-mass example with five degree of freedom with the following target set of eigenvalues:

$$\lambda^* = diag(-0.095 \pm 1.4i, -0.057 \pm 1.3i, -0.055 \pm 0.83i, -0.018 \pm 0.62i, -0.0024 \pm 0.2i)$$

and an initial approximation (α^0, β^0) given by:

$$\alpha^0 = (1, 1, 1, 1, 1), \quad \beta^0 = (1, 1, 1, 1, 1).$$

The results obtained by Newton's method are displayed on the left and those by the hybrid method on the right of Fig. 3. *These results clearly show that the convergence of the hybrid method is faster than the Newton method.*

7.2. Comparison of the proposed Newton method with the Elhay–Ram method [12]

Finally, we compare the Elhay–Ram method [12] with our proposed Newton's method. The same five degree-of-freedom mass-spring system example is used with the same target set and initial approximations as above. From Fig. 4 it is seen that *the proposed Newton method requires lesser number of iterations and the convergence behavior is much smoother than the Elhay–Ram method.*

8. Conclusion

In this paper we have proposed three numerical methods and developed associated mathematical theories for solving affine quadratic inverse eigenvalue problem (AQIEP). The proposed methods are: (i) A Newton's method, (ii) an alternating projections method, and (iii) a hybrid method combining Newton's method and the alternating projections method. A Newton method was also proposed earlier by Elhay and Ram in [13]. However, in [13], the authors assumed that the target eigenvalues and the eigenvectors for the current α and β iterates have the same number of real and complex pairs. Moreover, the authors did not consider ordering problem of the complex pairs of the eigenvalues.

Comparisons have been made amongst the proposed methods themselves and also with another Newton's method for AQIEP developed by Elhay and Ram [12]. The results of comparisons show that the hybrid method takes much lesser number of iterations and has better convergence behavior compared to the others, including the method of Elhay and Ram.

Our future work will be directed towards development of similar methods for the more challenging partially prescribed affine quadratic inverse eigenvalue problem (APQIEP). It is expected that the most of our theories developed for AQIEP will carry over to the solution of APQIEP, possibly with some necessary modifications.

Another important but difficult challenge of our future work will be to solve the AQIEP or APQIEP in such a way that resulting coefficient matrices C and K will have a prescribed structure. The proposed methods cannot guarantee any prescribed structure except for the symmetry.

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