ABSTRACT

QUADRATIC INVERSE EIGENVALUE PROBLEMS: THEORY, METHODS, AND APPLICATIONS

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This dissertation is devoted to the study of quadratic inverse eigenvalue problems from theoretical, computational and applications points of view. Special attention is given to two important practical engineering problems: finite element model updating and substructured quadratic inverse eigenvalue problems.

Because of their importance these problems have been well studied and there now exists a voluminous body of work, especially on finite element model updating, both by academic researchers and practicing engineers.

Unfortunately, many of the existing industrial techniques are ad hoc in nature and lack solid mathematical foundation and sophisticated state-of-the-art computational techniques. In this dissertation, some of the existing engineering techniques and industrial practices have been explained, whenever possible, by providing mathematical explanations with the help of new results on the underlying quadratic inverse eigenvalue problems, and based on these results, new techniques of model updating and substructured quadratic inverse eigenvalue problems have been proposed.

These results will contribute to advancement of the state-of-the-art knowledge in applied and computational mathematics, and mechanical vibrations and structural engineering. They will also impact the industries, such as automobile and aerospace companies, where these problems are routinely solved in their design and manufacturing.

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TABLE OF CONTENTS

Chapter **Page**

Chapter **Page**

LIST OF FIGURES

Figure **Page**

CHAPTER 1

Introduction

This dissertation deals with theory, computations and applications of *quadratic inverse eigenvalue problem* (QIEP). A QIEP is concerned with constructing three n-by-n matrices M, C, and K such that quadratic matrix pencil:

$$P(\lambda) = \lambda^2 M + \lambda C + K$$

has a prescribed eigenstructure and the matrices M, C, K satisfy certain constraints. By eigenstructure we mean a set of k eigenvalues and k eigenvectors and their multiplicities; k is not necessarily equal to $2n$. A special attention is given to two important QIEPs: finite element model updating (FEMU) and quadratic affine inverse eigenvalue problems. Both the problems have industrial applications and arise in vibration industries, such as automobile, air and space craft, buildings, bridges, and highways constructions, and others.

The vibrating structures are modeled by distributed parameter systems. In practice, such systems are discretized into a finite dimensional systems by using finite element techniques. The dynamics of finite element models are governed by the eigenvalues and eigenvectors of the pencil $P(\lambda)$. Usually, the finite element models are very large. Unfortunately, only a small number of eigenvalues and eigenvectors of the associated pencil are computable using the state-of-the-art computational techniques, such as *Jacobi-Davidson projection method*. Similarly only a small number

of them are also measurable experimentally, because of hardwire limitations. Under these circumstances, in order to numerically validate a finite element model, a vibration engineer must update an analytical finite element model using a small number of measured data from a prototype or a real-life structure. Finite element matrices usually have nice exploitable properties, such as the symmetry, positive definiteness, sparsity, etc., which are assets in a computational setting. Thus, the updating has to be done in such a way that the updated model reproduces the measured eigenvalues and eigenvectors and the original physical properties are preserved.

If a model has been updated this way, then the updated model can be used in confidence for future design and construction. Another important application is that an updated model is very often used to identify damages in structures, by comparing the physical parameters and differences of the elements of the original and updated models. Because of its industrial uses, the updating problem has been widely studied both by academic researchers and practicing engineers. As a result, a voluminous body of work exists.

There are two types of updating procedures. The first type of methods, assuming the mass matrix as the reference matrix, update first the measured data so that it satisfies the mass-orthogonality constraint (iii). This is then followed by updating the stiffness matrix so as to satisfy the constraints (ii) and (iv). The others update, either separately or simultaneously, the mass and stiffness matrices, satisfying the constraints (i)-(iv) [9, 42, 75]. There also now exists a method which updates the stiffness matrix first satisfying the constraints (ii) and (iv) and then computes the missing entries of the measured modes in a computational setting such that computed data satisfies the mass orthogonality constraint [10]. The method proposed in [10] has the additional important feature that the eigenvalues and eigenvectors

which are not updated remain unchanged by the updating procedure. This guarantees that "no spurious modes appear in the frequency range of interest."

In this dissertation, we propose a new method of the first type for a damped model. Thus, our method consists of two stages. In Stage I, we update the measured eigenvectors so that they satisfy a quadratic orthogonality relation proved in Corollary 5.1 of this dissertation. This result is a real-form generalization of the three orthogonality relations, proved earlier in [19]. In Stage II, the updated measured eigenvectors from Stage I are used to update the stiffness matrix so that it remains symmetric after updating and the measured eigenvalues and eigenvectors are reproduced by the updated model. Thus, our method generalizes methods for undamped models of the first type to a damped model. The results of numerical experiments on some case studies are presented to show the accuracy of the proposed method. Our contribution also includes mathematically established results to show that *satisfaction of the orthogonality relation by the measured data is necessary and sufficient for the solution of Stage II to be symmetric.*

It is to be noted that there are other methods for updating of a damped model. These include the method of Friswell, Inman and Pilkey [32], the algorithmic implementation of this method [44], and several control-theoretic methods (e.g, [56, 57, 76]). For details of the control theoretic methods, see Friswell and Mottershead [33, p. 154]. However, none of those methods explicitly updates the measured data so as to satisfy any orthogonality constraints. On the other hand, as noted before, our mathematical results of Theorems 5.6 and 5.7 demonstrate that Stage I must be performed explicitly or it is to be implicitly assumed that the measured data already satisfies an appropriate orthogonality constraint, before performing Stage II. Otherwise, the feasibility set of Stage II problem might be empty.

In our case the problems in both stages are nonlinear optimization problems. The Stage I problem is a nonconvex minimization problem with equality relations. This is a difficult optimization problem to solve. An augmented Lagrangian method is proposed to deal with this problem. Some convergence properties of this method are discussed.

The Stage II problem is a convex quadratic problem. This is a rather nice optimization problem to deal with and there are several excellent numerical methods for such problems in the literature (see [60]).

Implementations in optimization settings of Stage I and Stage II require that the appropriate gradient formulas must be computed in terms of the known quantities only, which are, in our case, just a few measured eigenvalues and eigenvectors and the corresponding sets from the analytical model. *Such gradient formulas have been mathematically derived in the dissertation.*

Besides the FEMU problem, another related QIEP, known as the *affine quadratic inverse eigenvalue problem* (AQIEP) is considered in this dissertation. The quadratic affine inverse eigenvalue problem is defined as follows:
Given a set of $2n$ self-conjugate scalars $\{\mu_1, ...\mu_{2n}\}$, the fixed matrix M, and a set of substructure matrices C_i's and K_i's, the quadratic affine inverse eigenvalue problem concerns with finding parameters α's and β's so that the pencil $P(\lambda) = \lambda^2 M + \lambda C(\alpha) + K(\beta)$ has $\{\mu_1, ...\mu_{2n}\}$ as its eigenvalues, where

$$C(\alpha) = \sum_{i=0}^{2n} \alpha_i C_i, \;\; K(\beta) = \sum_{i=1}^{2n} \beta_i K_i.$$

Such a substructure model arises, for example, when working with a free-free vibrating structure with finite number of degrees of freedom, in which each of the masses is connected by linear springs to other masses of the system.

Note that AQIEP can be view as FEMUP for a substructure model. Indeed, if we set $K = K_0$, $C = C_0$ and find the matrices $\Delta K,\ \Delta C$ in the form: $\Delta C = \sum_{i=1}^{2n} \alpha_i C_i$ and $\Delta K = \sum_{i=1}^{2n} \alpha_i K_i$, then $C_u = C + \Delta C,\ K_u = K + \Delta K$ could be found by solving affine QIEP. Affine models, for example, have been recently used in the analysis of helicopter structures [40]. The AQIEP also arises in model-aided diagnosis of mechanical systems [59]. Another example of application of the affine problem is constitutive equations, when decision about material properties is made by using the measured data [58].

In practical formulations of these problems it is usually assumed that the number of parameters is smaller then the number of measured eigenvalues; it is therefore more appropriate to consider the least square formulation of the problem, i.e, find α's and β's so that the spectrum of the pencil $(M, C(\alpha), K(\beta))$ is as close as possible to the set of target eigenvalues in the least square sense.

We propose new Newton-like and alternating projections algorithms in this dissertation to solve the AQIEP. These methods can be easily generalized to solution of the problem in least squares setting.

Here are the main contributions of this dissertation:

- A real-valued representation of the quadratic orthogonality result originally proved by Datta, et al. [19] is derived in Theorem 5.6. This result facilitates the use of existing optimization algorithms which are mostly formulated in real arithmetics.

- A new result (Theorem 5.7) on the QIEP is proved giving a necessary and sufficient condition on the existence and uniqueness of the matrix K, given the fixed matrices M and C, and a set of k self-conjugate scalars and vectors $(k < n)$, such that the spectrum of the pencil $P(\lambda)$ contains the given set of

scalars and the given vectors become the associated eigenvectors.

- Using the result cited above, a new algorithm is proposed for a damped model updating that satisfies the newly proved orthogonality relation. The above result also forms a mathematical basis for several model updating techniques, which implicitly assume such a result.

- New Newton-like and alternating projections algorithms are proposed for solution of the quadratic affine inverse eigenvalue problems.

- Convergence analysis of the proposed numerical algorithms are discussed in some details.

The new theoretical and computational results obtained in this dissertation will advance the state-of-the-art knowledge in computational and applied mathematics and vibration engineering. They are also likely to impact automobile and aircraft and other vibration industries which routinely solve the finite element model updating problem. I also feel very strongly that my own interdisciplinary training blending applied and computational mathematics with vibration engineering during the development of this dissertation is an asset for my future career.

Here is an outline of the dissertation:

Chapter 2 covers the basic results on quadratic eigenvalue problem, including important orthogonality relations and the notion of real-valued representation of the eigenvalues and eigenvectors.

Chapter 3 gives an introduction to inverse eigenvalue problems associated with quadratic matrix pencils. The inverse problems are grouped into several categories according to their structure-preserving natures, and for each of the categories recent results on those problems are reviewed.

Chapter 4 provides necessary background on optimization techniques which are used in the subsequent discussions of the optimization-based algorithm of the problems discussed in this dissertation.

Chapter 5 describes theory and computations of the finite element model updating problem. Sections 5.2 and 5.3 provide a review of model updating techniques for damped and undamped ($C = 0$) models, respectively. A new two-stage approach for model updating for a damped model is described in Section 5.4. A new result on QIEP that provides a mathematical justification of our formulation of Stage II is also proved in this section. A new algorithm, based on the two-stage approach, along with its convergence analysis and results on numerical experiments are presented in Section 5.5. Most of the research presented in this chapter is the result of joint work with Daniil Sarkissian and Sien Deng [26].

The affine inverse eigenvalue problem is described in Chapter 6. A locally quadratically convergent Newton-like algorithm is proposed in Section 6.2. Then two globally convergent algorithms based on alternating projections approach are presented in section 6.4. A matrix nearness problem, which arises as an auxiliary problem while computing projection operators necessary for applying alternating projections, is considered in Section 6.3. A combined usage of both locally convergent and globally convergent algorithms proposed in the Sections 6.2 and 6.3 is considered in Section 6.5. The last section contains some illustrative numerical experiments.

CHAPTER 2

Quadratic Eigenvalue Problem

2.1 Definition and Basic Results

Given matrices M, C and K, each of order n, the matrix

$$P(\lambda) = \lambda^2 M + \lambda C + K \tag{2.1.1}$$

is called the *quadratic matrix pencil*. For convenience, quadratic pencils will be denoted by the symbol (M, C, K). In vibration design and analysis the matrices M, C and K are called, respectively, the **mass matrix**, **damping matrix** and **stiffness matrix**. *These names describe the nature of the matrices. However, historically these names are used regardless of the application.* We assume that they are real.

The matrix pencil is called **singular** if $\det P(\lambda) = 0$, for all values of λ; otherwise it is called **regular**. *Unless otherwise is stated, we will assume that the pencil is regular.*

A scalar λ and an n-vector ϕ such that

$$P(\lambda)\phi = 0 \tag{2.1.2}$$

are, respectively, called an *eigenvalue*, and *eigenvector* of $P(\lambda)$. The eigenvalues are the roots of the polynomial equation

$$\det P(\lambda) = 0,$$

known as the **characteristic equation**. If M is nonsingular, then the there are $2n$ eigenvalues and eigenvectors. The problem of finding the eigenvalues and eigenvectors of $P(\lambda)$ is known as the **quadratic eigenvalue problem (QEP).**

The underlying equation, which is often used in dynamic analysis of mechanical systems, is a homogeneous linear second-order differential equation:

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0. \tag{2.1.3}$$

Mechanical structures are usually modeled by the equations, which are typically obtained by finite element discretization of distributed parameter systems.

Using separation of variables and assuming a solution of the form $q(t) = \phi_0 e^{\lambda_0 t}$, the equation (2.1.3) leads us to the eigenvalue-eigenvector problem:

$$P(\lambda_0)\phi_0 = 0.$$

In the case, when all of the eigenvalues of the quadratic pencil are distinct, the general solution to the above equation (2.1.3) is:

$$q(t) = \sum_{k=1}^{2n} a_i \phi_i e^{\lambda_i t}.$$

More generally, when λ_0 is an eigenvalues of algebraic multiplicity p, function

$$q(t) = \left(\frac{t^k}{k!}\phi_0 + ... + \frac{t}{1!}\phi_{k-1} + \phi_p \right) e^{\lambda_0 t}$$

is a solution of the differential equation if the set of vectors $\phi_0, ..., \phi_p$, with $\phi_0 \neq 0$, satisfies the relation

$$\sum_{p=0}^{j} \frac{1}{p!} L^{(p)}(\lambda_0)\phi_{j-p} = 0, \ j = 1, ..., p.$$

Here $L^{(p)}$ is the pth derivative of the polynomial. Such set of vectors $\{\phi_1, ..., \phi_p\}$ is called a *Jordan chain* of length $p + 1$ associated with eigenvalue λ_0. The Jordan

matrix of the pencil is the matrix in Jordan canonical form of size $2n$ that contains all information about the eigenvalues and their multiplicities. It is a generalization of the standard Jordan canonical matrix of a matrix A. If the geometric multiplicity of each of the eigenvalues is one, then the pencil is called **semisimple.** It means that all of the Jordan chains are of length one and consist of the corresponding eigenvectors.

It is convenient to organize eigenvalues and eigenvectors into matrices, as follows:

$$\Phi = (\phi_1, ..., \phi_k),\ \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix}. \tag{2.1.4}$$

Definition 2.1 *The matrices Λ and Φ are called eigenvalue and eigenvector matrices respectively. The pair (Φ, Λ) will be called k-matrix eigenpair, or just matrix eigenpair.*

Note, that the matrix eigenpair (Φ, Λ) of the pencil $P(\lambda)$ satisfies the following matrix equation:

$$M\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = 0. \tag{2.1.5}$$

2.2 Applications

The QEP arises in a wide variety of applications, including mechanical vibrations and structural engineering, physics, acoustic studies, etc., (see [73]). As noted before, mechanical vibrating systems are very often modeled by a system of second-order differential equations of the form (2.1.3). Very often in practical applications, these matrices have special structural properties. To see this, consider the following mass-spring system:

Figure 2.1 illustrates a simple mass-spring system with consequently connected masses $\{m_1, ..., m_n\}$. The resistance to displacement is provided by springs with

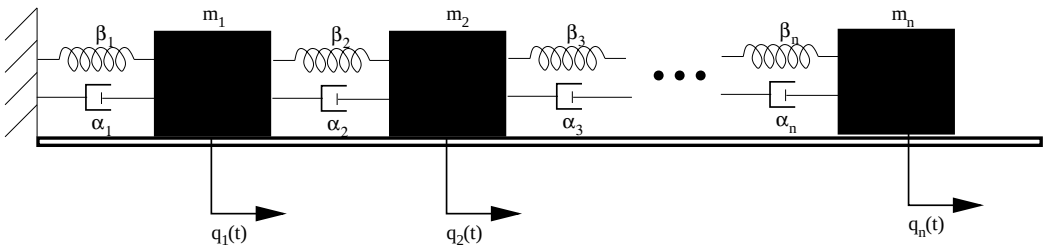

Figure 2.1: Mass-spring system.

stiffness constants $\{\beta_1, ..., \beta_n\}$, respectively, and the energy dissipation mechanism is represented by dampers with coefficients $\{\alpha_1, ..., \alpha_n\}$. Assuming that the damping is proportional to velocity $\dot{q}_i(t)$, the free vibrations (with no external forces) of the mass-spring system will be governed by the system of the form (2.1.3) with matrices M, C, and K given by

$$
\begin{aligned}
M &= \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix}, \\
C &= \begin{pmatrix} \alpha_1+\alpha_2 & -\alpha_2 & & \\ -\alpha_2 & \alpha_2+\alpha_3 & -\alpha_3 & \\ & \ddots & \ddots & \\ & & -\alpha_n & \alpha_n \end{pmatrix}, \\
K &= \begin{pmatrix} \beta_1+\beta_2 & -\beta_2 & & \\ -\beta_2 & \beta_2+\beta_3 & -\beta_3 & \\ & \ddots & \ddots & \\ & & -\beta_n & \beta_n \end{pmatrix}.
\end{aligned}
\tag{2.2.6}
$$

The **natural frequencies** are related to the eigenvalues of the associated quadratic matrix pencil and the eigenvectors are called **mode shapes** or just **modes**.

Note that coefficient matrices M, C and K of the above system are structured; moreover, all three of them are symmetric positive definite. Indeed, the matrices arising from the vibrating structures often have nice physical properties, such as:

- M is symmetric positive definite ($M = M^T > 0$) and often diagonal or tridi-

agonal

- C is symmetric semi-definite ($C = C^T \geq 0$)
- K is symmetric semi-definite ($K = K^T \geq 0$) and often tridiagonal or banded.

Unless otherwise stated we will assume throughout this dissertation that

- M is symmetric positive definite ($M = M^T > 0$)
- C is symmetric semi-definite ($C = C^T \geq 0$)
- K is symmetric semi-definite ($K = K^T \geq 0$).

The fact that the damping matrix is semidefinite corresponds to the fact that energy either dissipates or doesn't change due to the damping. For the mass-spring system, energy always dissipates due to nonzero damping, thus matrix C is positive definite, $C > 0$. Forces of the type $H\dot{q}(t)$, where H is skew-symmetric, can also be found in mechanical applications. They are of a different nature and are called the *gyroscopic forces*. The gyroscopic systems will not be considered in this dissertation.

Damping is present in all mechanical systems; however, it is, in general, hard to estimate. For the sake of computational convenience, it is very often assumed, that the damping matrix $C = 0$ or, more generally, is proportional to the mass and stiffness matrices. That is, C is assumed to be of the form:

$$C = \sum_j a_j M(M^{-1}K)^j.$$

The Rayleigh damping, where $C = a_0 M + a_1 K$, is a particular case of proportional damping, which is very often used for convenience. The assumption of proportional damping makes analysis much more simple; however, this assumption is not valid for many real-life applications. For example, systems in which gyroscopic forces

are presented are nonproportionally damped. One of the classical examples of a gyroscopic force, is the **Coriolis force** [53]. The Coriolis force arises in equations of relative motion, which are written in a moving coordinate system, that rotates with angle velocity.

2.3 Orthogonality Relation

Recall a classical result, which states that eigenvector matrix Φ of the symmetric generalized eigenvalue problem $K - \lambda M$, provided $M > 0, K \geq 0$ can be scaled such that $\Phi^T M \Phi = I$ and $\Phi^T K \Phi = \Lambda$, where Λ is the eigenvalue matrix [18].

A generalization for the quadratic matrix polynomial case has been obtained by Datta, Elhay, and Ram [19], and Lancaster [46].

Theorem 2.1 *Let $P(\lambda) = \lambda^2 M + \lambda D + K$, where $M = M^T > 0, C = C^T$, and $K = K^T$. Assume that the eigenvalues $\lambda_1, ..., \lambda_{2n}$ are all distinct and different from zero. Let $\Lambda = diag(\lambda_1, ..., \lambda_{2n})$ be the eigenvalue matrix and $\Phi = (\phi_1, ..., \phi_{2n})$ be corresponding matrix of eigenvectors. Then there exist diagonal matrices D_1, D_2, and D_3 such that*

$$\Lambda \Phi^T M \Phi \Lambda - \Phi^T K \Phi = D_1 \tag{2.3.7}$$

$$\Lambda \Phi^T C \Phi \Lambda + \Lambda \Phi^T K \Phi + \Phi^T K \Phi \Lambda = D_2 \tag{2.3.8}$$

$$\Lambda \Phi^T M \Phi + \Phi^T M \Phi \Lambda + \Phi^T C \Phi = D_3. \tag{2.3.9}$$

Furthermore,

$$D_1 = D_3 \Lambda;\ D_2 = -D_1 \Lambda; D_2 = -D_3 \Lambda^2.$$

Note that the matrices Λ and Φ are in general complex. Thus, Λ^T and Φ^T, are, respectively the transposes of the complex matrices. There now exists real representation of these orthogonality relations, which will be described in Section 2.5.

2.4 Numerical Methods

The QEP is a special case of the general polynomial eigenvalue problem. A classical reference book for the polynomial eigenvalue problem is the book by Lancaster [46]. A more updated account can be found in the book by Gohberg, Lancaster, Rodman [36]. The numerical methods for the polynomial eigenvalue problems are not well developed. Indeed, even in the case of the QEP, the state-of-the art computational techniques, such as the Jacobi-Davidson method [71], can compute only a few extremal eigenvalues and eigenvectors. A good account of the numerical techniques for the QEP can be found in the recent paper by Tisseur and Meerbergen [73]. We will now briefly review some of these techniques.

One of the standard approaches to investigate polynomial eigenvalue problem is to use linearizations [55]. By the linearization, it is meant, the reduction of the quadratic matrix pencil to a larger linear matrix pencil, which is "equivalent" to the given matrix polynomial $P(\lambda)$. There are infinitely many linear matrix pencils $A-\lambda B$, which are equivalent to the quadratic pencil (M, C, K). The size of matrices A, B, is necessarily $2n$. A quadratic pencil $P(\lambda)$ is equivalent to a linear matrix $A - \lambda B$ if

$$A - \lambda B = E(\lambda) \begin{pmatrix} P(\lambda) & 0 \\ 0 & I \end{pmatrix} F(\lambda)$$

for some $2n \times 2n$ matrix polynomials $E(\lambda)$ and $F(\lambda)$ with constant nonzero determinants; $P(\lambda)$ and $A - \lambda B$ have the same eigenvalues. The linear pencil $A - \lambda B$ is called linearization of $P(\lambda)$.

Companion form linearizations are the most widely used in the literature. A most commonly used companion form linearization is:

$$\begin{pmatrix} 0 & W \\ C & K \end{pmatrix} - \lambda \begin{pmatrix} W & 0 \\ 0 & -M \end{pmatrix}, \tag{2.4.10}$$

where W is an arbitrary nonsingular matrix. If matrix M is invertible, we can have the following linearization:

$$C_R - \lambda I = \begin{pmatrix} 0 & I \\ -M^{-1}C & -M^{-1}K \end{pmatrix} - \lambda I,\ C_L - \lambda I = \begin{pmatrix} 0 & -KM^{-1} \\ I & -CM^{-1} \end{pmatrix} - \lambda I \quad (2.4.11)$$

Unfortunately, the above companion form linearizations do not preserve any structure, of the original quadratic pencil. A useful linearization, which preserves symmetry is:

$$A - \lambda B = \begin{pmatrix} -K & 0 \\ 0 & W \end{pmatrix} - \lambda \begin{pmatrix} C & M \\ W & 0 \end{pmatrix}. \quad (2.4.12)$$

Note, that this linearization is symmetric, when $W = M$.

The linearization given in (2.4.11) and the symmetric linearization given in (2.4.12) are related as:

$$C_R = B^{-1}A.$$

Note, that if (Φ, Λ) is the matrix eigenpair of the pencil (M, C, K), then the linearized pencils, mentioned above, would have

$$\left(\begin{pmatrix} \Phi \\ \Phi\Lambda \end{pmatrix}, \Lambda \right)$$

as the matrix eigenpair. As was mentioned above, the matrix eigenpair (Φ, Λ) satisfies the so-called eigenvalue-eigenvector relation (2.1.5).

Computing the Eigenvalues of the QEP

Once the QEP is transformed to a linearized form, the resulting generalized eigenvalue problem can be solved by applying QZ iterations [18, 37]. The eigenvectors of $A - \lambda B$ are computed by using the generalized inverse iteration, and the eigenvectors of the quadratic pencil can be extracted from those of $A - \lambda B$.

There are also projection methods developed for finding a few eigenpairs of a quadratic pencil. These methods are used when the problem is rather sparse and

large. The Jacobi-Davidson method is a widely known representative of the family of projection methods. Specifically, the Jacobi-Davidson method can be briefly escribed as follows:

The Jacobi-Davidson Method [70]

- Compute an eigenpair $(\widetilde{\lambda}, \widetilde{\phi})$ of the projected problem $V_k^T P(\lambda) V_k$, corresponding to an eigenpair (λ, ϕ) of $P(\lambda) = \lambda^2 M + \lambda C + K$ with $\phi^* \phi = 1$, by finding an orthonormal basis $V = \{v_1, ..., v_k\}$ for the Krylov subspace $\mathcal{K}_k$. The pair $(\widetilde{\lambda}, \widetilde{\phi})$ is called a Ritz pair.

- Compute the correction pair v, η by solving the linear system:
$$\begin{pmatrix} P_2(\widetilde{\lambda}) & P_2'(\widetilde{\lambda})\widetilde{\phi} \\ 2\widetilde{\phi} & 0 \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} -r \\ 0 \end{pmatrix},$$
where $P_2(\widetilde{\lambda}) = \widetilde{\lambda}^2 M + \widetilde{\lambda} C + K$, $P_2'(\widetilde{\lambda}) = 2\widetilde{\lambda} M + C$, and $r = P_2(\widetilde{\lambda})\widetilde{\phi}$ is the residual.

- Obtain the new basis vectors v_{k+1} by orthogonalizing v against the previous column of the orthogonal basis matrix $V_k = (v_1, ..., v_k)$.

- Repeat until $||P(\widetilde{\lambda})\widetilde{\phi}||$ is small.

For more information on numerical solution of QEP (see [70]).

2.5 Real-Valued Representation for the Eigenvalues and Eigenvectors of the QEP

The matrix eigenpair (Φ, Λ), is in general complex for the QEP. However, from computational point of view it is more convenient to deal with the real-form representation of the pair (Φ, Λ). Later in this dissertation optimization problems will be considered, which will have eigenvector matrix as a variable. The MATLAB optimization toolbox, is used to solve some of these problems. One of the drawbacks

of the routines implemented in the toolbox is that only real-valued variables can be used. Usage of real matrix eigenpair, helps to overcome this drawback; moreover, it significantly simplifies derivation of the gradient formulas, necessary to implement optimization algorithms. We will show now how this can be done. We first note the fact that when the pencil is **semisimple**, i.e., all the eigenvalues have geometric multiplicity one, then the matrix

$$col(\Phi, \Phi\Lambda) = \begin{pmatrix} \Phi \\ \Phi\Lambda \end{pmatrix} \text{is of full rank.} \tag{2.5.13}$$

See [36].

First, we note, that conditions (2.1.5) and (2.5.13) will hold if we replace the k-matrix eigenpair (Φ, Λ) by $(\Phi S^{-1}, S\Lambda S^{-1})$, for some invertible $2k \times 2k$ $(k \leq n)$ matrix S. Let the eigenvalue matrix of $P(\lambda)$ be $\Lambda = diag(\alpha_1 + i\beta_1, \alpha_1 - i\beta_1, ..., \alpha_l + i\beta_l, \alpha_l - i\beta_l, \lambda_{2l+1}, ..., \lambda_k)$ and the corresponding eigenvector matrix be $\Phi = (u_1 + iv_1, u_1 - iv_1, ..., u_l + iv_l, u_l - iv_l, \phi_{2l+1}, ..., \phi_k)$. Define the pair $(X, T) = (\Phi S^{-1}, S\Lambda S^{-1})$, with

$$\begin{gathered} S = diag(S_1, ..., S_l, S_{2l+1}, ..., S_k), \text{ where,} \\ S_j = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \ j = 1, ..., l \\ 1, \ j = 2l+1, ..., k \end{cases} \end{gathered} \tag{2.5.14}$$

Each two by two blocks of the matrix S corresponds to a complex conjugate pairs of eigenvalues of $P(\lambda)$. The pair (X, T) will be called **real matrix eigenpair** of $P(\lambda)$. Note, that the matrices X and T will have the following structure:

$$X = [u_1, v_1, ..., u_l, v_l, \phi_{2l+1}, ..., \phi_k] \tag{2.5.15}$$

$$\begin{gathered} T = diag(T_1, ..., T_l, T_{2l+1}, ..., T_k), \text{ with} \\ T_j = \begin{cases} \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}, \ j = 1, ..., l \\ \lambda_j, \ j = 2l+1, ..., k \end{cases} \end{gathered} \tag{2.5.16}$$

Definition 2.2 *The pair* (X, T), $X \in \mathbb{R}^{k\times 2k}$ *and* $T \in \mathbb{R}^{2k\times 2k}$, *as defined above, will be called a k-real matrix eigenpair* ($k \leq 2n$) *or just a real matrix eigenpair.*

The real matrix eigenpair (X, T) satisfies the following matrix equations:

$$MXT^2 + CXT + KX = 0. \tag{2.5.17}$$

Assuming that the eigenvalues are semisimple, we have,

$$\begin{pmatrix} X \\ XT \end{pmatrix} \text{ is of full rank.} \tag{2.5.18}$$

CHAPTER 3

Quadratic Inverse Eigenvalue Problem [QIEP]

3.1 Statement

Given a partial or complete eigenstructure of a matrix A, the problem of computing the matrix from the given eigenstructure is called the *inverse eigenvalue problem.*

Quadratic Inverse Eigenvalue Problem: The inverse eigenvalue problem for the quadratic pencil $P(\lambda)$ is similarly defended. We will denote the problem by the symbol *QIEP*. Specifically, given a partial or complete eigenvalue-eigenvector information, the QIEP is to find the matrices M, C and K, such that quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ has a prescribed eigenstructure.

The inverse eigenvalue problems are as equally important as the direct eigenvalue problems. Indeed, as we will see in this dissertation, that many practical applications give rise to QIEP. For an account of theory and applications of the inverse eigenvalue problems, see the book by Chu and Golub [15].

The QIEP arising in finite element model updating problem, which is considered in this dissertation, concerns modifying the matrices M, C, K from knowledge of only the partial spectrum and the associated eigenvectors. We will briefly review some relevant results on this topic now.

3.2 Unstructured QIEP from Fully Prescribed Eigenstructure

Given a set of $2n$ scalars $\{\lambda_1, ..., \lambda_{2n}\}$ and $2n$ n-vectors $\{\phi_1, ..., \phi_{2n}\}$, the unstructured QIEP is the problem of finding three matrices M, C, and K, not necessarily symmetric or having any specific properties, such that the spectrum of the quadratic pencil

$$P(\lambda) = \lambda^2 M + \lambda C + K$$

will include the numbers $\{\lambda_1, ..., \lambda_{2n}\}$ as its eigenvalues and $\{\phi_1, ..., \phi_{2n}\}$ will be the corresponding eigenvectors. The problem is rather easy to solve.

Define the matrices Λ and Φ as in (2.1.4).

Theorem 3.1 ([36]) *Assume that matrix $col(\Phi, \Phi\Lambda)$ is of full rank. Then solution to the QIEP can be found as follows:*

$$\begin{aligned} &M \text{ – arbitrary} \\ &C = -M\Phi\Lambda^2 Q_2 \\ &K = -M\Phi\Lambda^2 Q_1, \end{aligned} \tag{3.2.1}$$

where Q is such that

$$Q col(\Phi, \Phi\Lambda) = I,$$

and $Q = (Q_1, Q_2)$, $Q_i \in \mathbb{R}^{k\times n}$.

Proof. It is easy to see that

$$M\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = M\Phi\Lambda^2(I - Q col(\Phi, \Phi\Lambda)) = 0$$

□

In his work Lancaster uses a more general notion of matrix eigenpair, the so-called standard pairs and triples [36].

Definition 3.1 *A pair of matrices* (U, D), $U \in \mathbb{C}^{n\times 2n}$, $D \in \mathbb{C}^{2n\times 2n}$ *is called a standard pair if there exists nonsingular matrix* S, *such that* $U = \Phi S^{-1}$, $D = S\Lambda S^{-1}$, *where* (Φ, Λ) *is a matrix eigenpair.*

For example, a real matrix eigenpair is a particular case of a standard pair.

Definition 3.2 *Three matrices* U, D, V, $U \in \mathbb{C}^{n\times 2n}$, $D \in \mathbb{C}^{2n\times 2n}$ *and* $V \in \mathbb{C}^{2n\times n}$ *form a standard triple if* (U, D) *is a standard pair, and* $UV = 0$, $UDV = M^{-1}$.

Thus, the matrices U and V generalize the notion of matrices of right and left eigenvectors. It is easy to show that Theorem 3.1 holds for a standard pair. Thus, given a nonsingular matrix M and a standard pair (U, D), the relation (3.2.1) uniquely determines the matrices C, K of the pencil (M, C, K). In other words, we can say that a standard triple (U, D, V) uniquely defines the associated pencil. The matrices (M, C, K) can also be defined in terms of *moments* [49], defined by:

$$\Gamma_j = UD^jV,\ j = 0, 1, ... \tag{3.2.2}$$

The moments are invariant under the choice of a triple.

Theorem 3.2 *Let* (U, D, V) *be a standard triple and define the moments by (3.2.2). Then the corresponding unique pencil* (M, C, K) *can be constructed as follows:*

$$M = \Gamma_1^{-1},\ C = -M\Gamma_2 M,\ K = -M\Gamma_3 M + C\Gamma_1 C. \tag{3.2.3}$$

Proof. Note, formulas (3.2.1) obtained above will hold for a standard pair (U, D) as well: i.e., pencil $(M, -MUD^2Q_2, -MUD^2Q_1)$ is the pencil associated with standard pair (U, D), where M is arbitrary and $col(U, UD)(Q_1, Q_2) = I$.

Now we need to show that $C = -M\Gamma_2 M = -MUD^2VM = -MUD^2Q_2$ and $K = -M\Gamma_3 M + C\Gamma_1 C = -MUD^2Q_1$. It is easy to see, that $VM = Q_2$. Indeed,

$$\begin{pmatrix} U \\ UD \end{pmatrix} (Q_1, Q_2) = \begin{pmatrix} UQ_1 & UQ_2 \\ UDQ_1 & UDQ_2 \end{pmatrix} = I.$$

Thus, $UDQ_2 = I$. Postmultiply by M^{-1}, $UDQ_2M^{-1} = M^{-1} = UDV$, and obtain $VM = Q_2$, which shows that $C = -M\Gamma_2 M$.

Now, lets consider formula for the matrix $K = -M\Gamma_3 M + C\Gamma_1 C = -MUD^2Q_1$. Rearranging this expression we have

$$-MUD^2(DVM - Q_2UD^2Q_2) = MUD^2Q_1;$$

i.e., it is enough to show that $DVM - Q_2UD^2Q_2 = Q_1$. Premultiplying this expression by U, we obtain $UDVM = UQ_1 = I$ and this completes the proof. □

Consider triple of matrices (Φ, Λ, Y), where (Φ, Λ) is an eigenpair and $2n \times n$ matrix Y satisfies

$$\begin{pmatrix} \Phi \\ \Phi\Lambda \end{pmatrix} Y = \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix}. \tag{3.2.4}$$

Note, that (Φ, Λ, Y) is just a particular case of the standard triple. Thus, the result of Theorem 3.2 will hold for the triple (Φ, Λ, Y).

Corollary 3.1 *Given matrix eigenpair (Φ, Λ) and nonsingular matrix M, let Y be as defined above in (3.2.4). Define the moments by*

$$\Gamma_j = \Phi\Lambda^j Y, \ j = 0, 1, ... \tag{3.2.5}$$

Then corresponding unique pencil (M, C, K) can be constructed as follows:

$$M = \Gamma_1^{-1}, \ C = -M\Gamma_2 M, \ K = -M\Gamma_3 M + C\Gamma_1 C. \tag{3.2.6}$$

In the previous section it was shown that quadratic eigenvalue problem is equivalent to a standard or generalized eigenvalue problem of dimension $2n$. Thus, well-studied techniques for the standard or the generalized eigenvalue problem can be used to obtain solution of the quadratic problem. However, this is not the case for inverse problems. It is much harder to use algorithms for standard inverse eigenvalue problems to solve quadratic inverse eigenvalue problem. Lancaster and Prells [63] have studied *structure preserving equivalences* and *structure preserving similarities*. The idea is to start with one system, say (M_0, D_0, K_0), and, implicitly, a complete set of corresponding spectral data, and to show how to generate isospectral pencils (M, D, K) which share the same set of eigenvalues (including their multiplicity structures). And this is to be done without explicit reference to eigenvalues and eigenvectors.

Definition 3.3 *Let $A - \lambda B$ be a symmetric linearization of the pencil (2.1.1) as in (2.4.12). Let $A' - \lambda B'$ be obtained from $A - \lambda B$ by strict equivalence, i.e., $A' = EAF$, $B' = EBF$ for some nonsingular E and F. Then this strict equivalence is said to be **structure preserving** if A' and B' have the same form as A and B, i.e.:*

$$A' = \begin{pmatrix} -K' & 0 \\ 0 & M' \end{pmatrix}, \; B' = \begin{pmatrix} C' & M' \\ M' & 0 \end{pmatrix} \tag{3.2.7}$$

and M' is nonsingular.

In this case $A' - \lambda B'$ is a linearization of the pencil M', C', K'. Observe, that structure preserving equivalence (SPE) could be obtained from a transformation of $P(\lambda)$. If S and H are nonsingular, then transformation $SP(\lambda)H$ corresponds to an SPE with

$$E = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \; F = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}.$$

Definition 3.4 *Let C_R be the right companion matrix (2.4.11) of $P(\lambda)$. The nonsingular $2n \times 2n$ matrix S_R is called a (right) structure preserving similarity (SPS) if $S_R^{-1} C_R S_R$ is a (right) companion matrix.*

These transformations are closely related.

Theorem 3.3 *Two quadratic pencils are related by an SPE if and only if they are related by an SPS.*

Proof. Let $A - \lambda B = E(A_0 - \lambda B_0)F$, then F defines right structure preserving similarity. Indeed,

$$B^{-1}A = F^{-1}B_0^{-1}E^{-1}EA_0F = F^{-1}B_0^{-1}A_0F$$

Conversely, let S be a structure preserving similarity, then

$$A - \lambda B = B(C_R - \lambda I) = B(S^{-1}C_R^{(0)}S - \lambda I) = \\ BS^{-1}(C_R^{(0)} - \lambda I)S = BS^{-1}B_0^{-1}(A_0 - \lambda B_0)S.$$

□

In their paper [63], Prells and Lancaster, obtained an expression for an structure preserving equivalence.

Theorem 3.4 *Let $P_0(\lambda)$ be given matrix pencil and $A_0 - \lambda B_0$ be its symmetric linearization (2.4.12), and write $C_0 = B_0^{-1}A$. Consider triple (X, C_0, Y), s.t $col(C_0, C_0X)$ is nonsingular, $XY = 0$ and XC_0V is nonsingular.*

Then the $2n \times 2n$ matrices $(B_0Y \ A_0Y)$ and $\begin{pmatrix} X \\ XC_0 \end{pmatrix}$ are nonsingular and

$$E := (Y \ C_0Y)^{-1}B_0^{-1}, \ F := \begin{pmatrix} X \\ XC_0 \end{pmatrix}^{-1}$$

determine an SPE of the given pencil.

Conversely, every pencil $P(\lambda)$ isospectral with $P_0(\lambda)$ can be obtained from some choice of X and Y in a standard triple (X, C_0, Y).

Proposition 3.1 *Let X, C_0, Y be as defined in the theorem above. Then the corresponding pencil has Hermitian coefficients if $Y = B_0^{-1}X^*$.*

The condition $X^* = B_0 Y$ can be rewritten in the following way:

$$X^* = B_0 \begin{pmatrix} X \\ XC_0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix},$$

and

$$\begin{pmatrix} X \\ XC_0 \end{pmatrix} B_0^{-1} X^* = \begin{pmatrix} O \\ M^{-1} \end{pmatrix}.$$

Thus, Hermitian coefficients are generated provided the two following conditions are satisfied:

$$XB_0^{-1}X^* = 0,\ XC_0B_0^{-1}X^* = M^{-1}.$$

Even though the results presented above give some insight into the problems, from computational point of view it is more viable to use matrix eigenpairs instead of triples (X, C_0, Y). Some computational schemes for inverse problem for pencils with non-real spectrum, based on matrix eigenpairs, were presented by Lancaster and Prells in [48]. The ideas introduced in [48] were then extended in [51], from the restriction to systems with purely nonreal spectrum to the full range of real and nonreal spectrum but with limitation to semisimple spectrum. *From now on it is assumed that all the eigenvalues are semisimple.* Since coefficient matrices are real, eigenvalues are real or appear in complex conjugate pairs. Assume there are $2r$ real eigenvalues ($0 \leq r \leq n$) and $2(n-r)$ complex eigenvalues $U_1 \pm iW$, with $W > 0$. Let real eigenvalues be distributed between two $r \times r$ real diagonal matrices U_2 and U_3, then the eigenvalues and eigenvectors matrices are now

$$\Lambda = \begin{pmatrix} U_1 + iW & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & U_1 - iW \end{pmatrix},\ \Phi = (\Phi_c, \Phi_{R1}, \Phi_{R2}, \overline{\Phi}_c). \tag{3.2.8}$$

Defining $\Omega_1^2 = U_1^2 + W^2$, it is easily seen that there is an associated pencil

$$P_0(\lambda) = \lambda^2 I - 2\lambda \begin{pmatrix} U_1 & 0 \\ 0 & \frac{1}{2}(U_2 + U_3) \end{pmatrix} + \begin{pmatrix} \Omega_1^2 & 0 \\ 0 & U_2 U_3 \end{pmatrix}, \tag{3.2.9}$$

which is a direct sum of two diagonal pencils:

$$\lambda^2 I_{n-r} - 2\lambda U_1 + \Omega_1^2 = (\lambda I_{n-r} - (U_1 + iW))(\lambda I_{n-r} - (U_1 - iW)) \tag{3.2.10}$$

and

$$\lambda^2 I_{2r} - \lambda(U_2 + U_3) + U_2 U_3 = (\lambda I_r - U_2)(\lambda I_r - U_3). \tag{3.2.11}$$

Define

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & \frac{1}{2}(U_2 + U_3) \end{pmatrix}, \text{ and } \Omega^2 = \begin{pmatrix} \Omega_1^2 & 0 \\ 0 & U_2 U_3 \end{pmatrix}$$

then (3.2.9) takes form

$$P_0(\lambda) = \lambda^2 I - 2\lambda U + \Omega^2. \tag{3.2.12}$$

The right companion matrix of this pencil is

$$C_0 = \begin{pmatrix} 0 & I \\ -\Omega^2 & 2U \end{pmatrix}. \tag{3.2.13}$$

Lemma 3.1 *Let the real eigenvalues be prescribed in such a way that*

$$\det(U_2 - U_3) \neq 0;$$

then the matrix Z, defined by

$$Z = \begin{pmatrix} U_1 - iW & 0 & 0 & -I \\ 0 & -U_3 & I & 0 \\ 0 & -U_2 & -I & 0 \\ U_1 + iW & 0 & 0 & I \end{pmatrix}, \tag{3.2.14}$$

is nonsingular and transforms the companion matrix C_0 into Λ, i.e.,

$$ZC_0Z^{-1} = \Lambda.$$

Using the matrix Z defined in the Lemma 3.1 it is possible to establish a one-to-one correspondence between matrix eigenpairs and structure preserving similarities.

Theorem 3.5 *Let Λ be a diagonal matrix of the form (3.2.8), for which* $\det(U_2 - U_3 \neq 0$.

(a) If Φ is a matrix of the for $\Phi = (\Phi_c \ \Phi_{R1} \ \Phi_{R2} \ \overline{\Phi}_c)$, where $\Phi_c \in \mathbb{C}^{n\times n-r}$ and $\Phi_{R1},\ \Phi_{R2} \in \mathbb{R}^{r\times r}$, i.e., (Φ, Λ) form a matrix eigenpair, then with Z defined by (3.2.14),

$$V = \begin{pmatrix} \Phi \\ \Phi\Lambda \end{pmatrix} Z \tag{3.2.15}$$

defines an SPS of C_0.

(b) Conversely, if V defines an SPS of C_0 and Z is defined by (3.2.14), then there exists a Φ in the form defined above, such that (6.3.14) holds.

Observe, the fact that eigenvalues and corresponding eigenvectors are real or presented in complex conjugate pairs will guarantee that the resulting isospectral family will have real coefficient matrices.

3.3 Structured QIEP with Fully Prescribed Eigenstructure

As we have just seen from the mass-spring example in the previous chapter, the matrices M, C and K have some nice structural properties (realness, symmetry, positive definiteness, bandness). It is a typical situation, when an application, which gives rise to the QIEP, assumes that the reconstructed matrices have some of the structural properties. Thus, it is required that the solution of the associated inverse eigenvalue problem satisfies the structural constraints. Thus, the structured QIEPs are the ones of practical interest. Finite element model updating (FEMU) problem, to be defined in the next section, is one such application problem. As we will see

later, the updated matrices must be symmetric, and positive definite (semidefinite) and must have specific sparsity pattern. Unfortunately, these requirements may not always be met - certain conditions must be satisfied. For example, the symmetry of matrices M, C, K implies that there are $\frac{2}{3}n(n+1)$ unknowns, but each eigenpair (ϕ, λ) characterizes a system of n equations. Thus, if a k-eigenpair (Φ, Λ) is given, with $k \leq \frac{2}{3}(n+1)$ (or $k \leq n+1$ if M is given), then there exists a symmetric solution to the inverse problem. In the case, $k > n+1$, a result giving a necessary and sufficient condition is presented in the following section. The existence of a solution, with the requirement that matrices are positive definite is a more complicated issue. The simple counting of variables and equations will not give us the answer. In [16] it was shown, that given a k-matrix eigenpair (Φ, Λ), with distinct eigenvalues and linearly independent eigenvectors, it is always possible to find real and symmetric matrices M and K solving QIEP with positive definite and semidefinite, respectively, provided $k \leq n$ (see [16]). Detail will be given later.

There now exist several results on reconstructing a structured quadratic matrix pencil from a spectral data [1, 2, 3, 13, 16, 21, 28, 30, 34, 35, 45, 47, 49, 51, 64, 65, 66, 72]. Solution to this type of problems typically is not unique. Below, several of the current results on this topic are summarized.

First, we consider the case of symmetric structure. It was shown in the previous section in Corollary 3.1 how matrices C, K can be constructed in terms of moments, when Φ, Λ and M are given. A more interesting question is how to reconstruct matrices C and K, which are symmetric.

Theorem 3.6 ([49]) *Let the system (M, C, K) be generated by triple (Φ, Λ, Y). Then the moments Γ_1, Γ_2 and Γ_3 (3.2.5) are all real, Hermitian, or real symmetric according as all of M, C, K are also real, Hermitian, or real symmetric repletely.*

It is now necessary to introduce an important notion of the sign characteristic associated with a real eigenvalue.

Definition 3.5 *Let (ϕ, λ) be a semisimple real eigenpair of the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$. Consider the following product:*

$$\phi^T P'(\lambda)\phi = 2\lambda\phi^T M\phi + \phi^T C\phi = \epsilon\kappa^2;$$

then $\epsilon = \pm 1$ defines the sign characteristic of λ.

Definition 3.6 *The eigenvector ϕ is called normalized, when $\kappa = 1$.*

Now assume that real eigenvalues in (3.2.8) are distributed is such a way that all eigenvalues with positive sign characteristic are presented in U_2 and with negative in U_3. Sign characteristic cannot be assigned arbitrarily; for the case of semisimple pencil we have

$$\sum_{j=1}^{2r} \epsilon_j = 0$$

(see [36]). Define permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & I_{n-r} \\ 0 & I_r & 0 & 0 \\ 0 & 0 & -I_r & 0 \\ I_{n-r} & 0 & 0 & 0 \end{pmatrix}. \tag{3.3.16}$$

Note that with Λ and Φ in the form (3.2.8), $(P\Lambda)^* = P\Lambda$ and $P\Phi^* = \Phi^T$. Lancaster and Prells [48] have proved the following results:

Theorem 3.7 *Given a nonsingular symmetric matrix M, the matrix eigenpair (Φ, Λ) (3.2.8) determines real symmetric matrices C and K if*

$$\begin{pmatrix} \Phi \\ \Phi\Lambda \end{pmatrix} P\Phi^* = \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix}.$$

Conversely, eigenvectors of a symmetric pencil can be normalized in such a way that the above relation holds.

Computational schemes of constructing a symmetric pencil, based on the result provided by the Theorem 3.7, have been investigated in [51].

Now if we assume that the spectral data is consistent with real and symmetric systems, the question is, how we can ensure the positive definiteness (semidefiniteness) of matrices (M, C, K)? To answer this question, observe that

$$-K^{-1} = X(\Lambda^{-1}P)X^* = \Gamma_{-1}. \tag{3.3.17}$$

This follows immediately from the resolvent form for $P(\lambda)$:

$$P(\lambda)^{-1} = \Phi(\lambda I - \lambda)^{-1}Y.$$

Theorem 3.8 *If Λ is stable (all eigenvalues lie in the open left half-plane), $\Gamma_2 \leq 0$, and $\Gamma_1,\ \Gamma_{-1}$ are nonsingular, and $M, C,$ and K are as defined in 3.1, then $M > 0,\ C \geq 0$, and $K > 0$.*

Proof. The stability of Λ implies that $M > 0$ and $K > 0$. Then $C \geq 0$ follows from $\Gamma_2 \leq 0$. □

3.4 Unstructured QIEP with Partially Prescribed Eigenstructure

In applications it is often desirable to reconstruct a matrix pencil when only a portion of the spectrum is given. For example, when the eigenvalues which need to be assigned come from a measurement data. One of the constraints that arises in practical applications is that for complicated large systems, only a few eigenvectors and eigenvalues, especially the smallest ones, can be computed. Similarly, because of hardware limitations, it is possibly only to measure in a laboratory or in an experiment with a real-life structure only a few eigenvalues and eigenvectors that correspond to the lowest natural frequencies of a vibrating structure. These

constraints limit us to consider QIEP which can be solved with partially prescribed eigenstructure. A few results now exist and we will state some of them in this section.

In a series of papers [19, 20, 22, 23, 24, 68], Datta, Ram, Elhay and Sarkissian in the last few years have solved several unstructured QIEP with partially prescribed spectrum. These problems naturally arise in controlling dangerous vibrations, such as resonance, in mechanical structures, including, buildings, bridges, highways, etc. These include, *quadratic partial eigenvalue assignment* and *quadratic partial eigenstructure assignment problems*, denoted, respectively, by *QPEVAP* and *QPEAP*. The QPEVAP concerns with re-assigning only a small part of the spectrum of the pencil (M, C, K) by using feedback control matrices F and G to another user-chosen set while keeping the remaining large part of the spectrum and the associated eigenvectors unchanged. The last phenomenon is known as the *no spill-over* in vibration engineering literature. The QPEAP is similarly defined. Here both a small part of the spectrum and the associated eigenvectors, are reassigned to user's chosen set of eigenvalues and eigenvectors by feedback control.

Mathematically the problems are defined as follows:

QPEVAP: Given the pencil (M, C, K); a set $\{\mu_1, ..., \mu_k\}$, closed under complex conjugation; and a matrix B, find matrices F and G, so that the pencil $(M, C - BF^T, K - BG^T)$ has $\{\mu_1, ..., \mu_k\}$ as its first k eigenvalues and the rest $2n - k$ eigenvalues and corresponding eigenvectors are the same as of the original pencil (M, C, K).

QPEAP: Given a pencil (M, C, K); a set $\{\mu_1, ..., \mu_k\}$ and set of vectors $\{y_1, ..., y_k\}$, both closed under complex conjugation; and a matrix B, find matrices F and G so that the pencil $(M, C - BF^T, K - BG^T)$ has $\{(\mu_1, y_1)..., (\mu_k, y_k)\}$ as its first k eigen-

pairs and the rest $2n-k$ eigenpairs are the same as of the original pencil (M, C, K).

A practical solution for QPEVAP [23] has been obtained in the quadratic setting without making any transformation to a standard first-order state-space form, thus, avoiding computation of the inverse of possibly ill-conditioned mass-matrix and the destruction of exploitable structural properties, including the symmetry, positive definiteness, boundness, sparsity, etc.

The other important features of the solution proposed in [23] are (i) that only the knowledge of the small number of eigenvalues and eigenvectors that need to be reassigned suffice for implementation, (ii) the no spill-over is established by means of mathematically proved results, and (iii) no model reduction is done, no matter how large the model may be.

We state the solution of QPEVAP here without any proof.

Theorem 3.9 *Let matrix B be of full rank. Let the scalars $\{\mu_1, ..., \mu_k\}$ and the eigenvalues of the pencil (M, C, K) be such that sets $\{\lambda_1, ..., \lambda_k\}$, $\{\lambda_{k+1}, ..., \lambda_{2n}\}$ and $\{\mu_1, ..., \mu_k\}$ are disjoint and each set is closed under complex conjugation. Let $Y = (y_1, ..., y_k)$ be the matrix of left eigenvectors associated with eigenvalues $\{\lambda_1, ..., \lambda_p\}$. Let the pair $(P(\lambda), B)$ be partially controllable with respect to $\{\lambda_1, ..., \lambda_k\}$, i.e., $y_i^* B \neq 0, i = 1, ..., k$. Let $\Gamma = (\gamma_1, ..., \gamma_k)$ be a matrix such that*

$$\gamma_j = \overline{\gamma}_i, \text{ whenever } \mu_j = \overline{\mu}_i.$$

Set $\Lambda_1 = diag(\lambda_1, ..., \lambda_k)$ and set $\Sigma = diag(\mu_1, ..., \mu_k)$. Let Z be the unique nonsingular solution of the Sylvester equation:

$$\Lambda_1 Z - Z\Sigma = -Y^* B\Gamma.$$

Let the real feedback matrices be given by

$$F = \Phi Y^* M, \text{ and } G = \Phi(\Lambda_1 Y^* M + Y^* C),$$

where Φ *satisfies the linear system* $\Phi Z = \Gamma$. *Then the closed-loop pencil* $(M, C - BF^T, K - BG^T)$ *will have* $\{\mu_1, ..., \mu_k, \lambda_{k+1}, ..., \lambda_{2n}\}$ *as its eigenvalues and the eigenvector corresponding to the eigenvalues* $\{\lambda_{k+1}, ..., \lambda_{2n}\}$ *will remain unchanged.*

Notes:

- The feedback matrices given by Theorem 3.9 are parameterized by the matrix Γ, then yielding a family of feedback matrices for different choices of Γ. Exploiting this parametric nature of the solution, Brahma and Datta [7, 8] have recently developed algorithms for minimum feedback norms and minimization of the condition number of the closed-loop eigenvector matrix. These algorithms constitute a numerically robust solution to QPEVAP.

- Similar expression for feedback matrices for QPEAP have been derived by Datta, Elhay, Ram, and Sarkissian [20].

3.5 Structured QIEP with Partially Prescribed Eigenstructure

In this section, we describe some results on structured QIEP which are also motivated by applications in vibration engineering.

The following theorem, proved by Chu, Kuo, and Lin [16], shows how to construct a symmetric pencil with mass and stiffness matrices being both positive definite and semidefinite, respectively, when partial eigenstructure is prescribed. Eigenstructure of the resulting pencil is also analyzed. Some attention is also paid to monic systems ($M = I$). Recall, that by (X, T) we denote real matrix eigenpair; i.e., T is the real-valued representation of the eigenvalue matrix and X is the real-valued representation of the eigenvector matrix. Instead of considering generalized inverse of $col(X, XT)$ to obtain the solution to the inverse problem as in (3.2.1), in [16]

the authors consider the null space of $col(X, XT)^T$, where (X, T) is a k-real matrix eigenpair, $X \in R^{2n\times k}$ and $T \in R^{k\times k}$, $(k \leq n)$.

Theorem 3.10 ([16]) *Let*

$$(Q_1\ Q_2)col(X, XT) = 0. \tag{3.5.18}$$

Then, a solution to the quadratic inverse eigenvalue problem, with matrices M and K being symmetric positive semidefinite, is given as follows:

$$\begin{aligned} M &= Q_2^T Q_2, \\ C &= Q_2^T Q_1 + Q_1^T Q_2, \\ K &= Q_1^T Q_1. \end{aligned} \tag{3.5.19}$$

It is shown in [16] that if X is of full column rank, then the resulting pencil (3.5.19) is regular, and singular otherwise. The remaining eigenstructure of the pencil is described in the following theorems.

Theorem 3.11 *Let (X, T) be a k-real matrix eigenpair and let the pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ be defined by (3.5.19). Assume, that X is of full column rank, then:*

1. *If $k = n$, then $P(\lambda)$ has double eigenvalue λ_j for each $\lambda_j \in \sigma(T)$.*

2. *If $k < n$, then $P(\lambda)$ has double eigenvalue λ_j for each $\lambda_j \in \sigma(T)$. The remaining $2(n-k)$ eigenvalues of $P(\lambda)$ are all complex conjugate with nonzero imaginary parts. In addition, if the matrices Q_1 and Q_2 are chosen from an orthogonal basis of the null space of $col(X, XT)^T$, then the remaining $2(n-k)$ eigenvalues are only $\pm i$ with corresponding eigenvectors $z \pm iz$, where $X^T z = 0$.*

Thus, when $k = n$ all eigenvalues are counted. It also possible to further calculate the geometric multiplicity of the double roots characterized in the previous theorem.

Theorem 3.12 *Let (X,T) be a k-real matrix eigenpair and let the pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ be defined by (3.5.19). Assume, that X is of full column rank, then:*

1. *Each real-valued $\lambda_j \in \sigma(T)$ has an elementary divisor of degree 2; that is, the dimension of the null space $\mathcal{N}(P(\lambda_j))$ is 1.*

2. *The dimension of $\mathcal{N}(P(\lambda_j))$ of a complex-valued eigenvalue $\lambda_j \in \sigma(T)$ is generically 1. That is, the pairs of matrices (X,T) of which a complex-valued eigenvalue has a linear elementary divisor form a measure zero set.*

Another numerical method for constructing matrices M, C and K, with M positive definite, when k-real matrix eigenpair (X,T) is given ($k \leq n$), is proposed in [45].

Theorem 3.13 ([45]) *Given a k-real matrix eigenpair (X,T), let*

$$X = Q \begin{pmatrix} R \\ 0 \end{pmatrix} \tag{3.5.20}$$

be the QR decomposition of X, and let $S = RTR^{-1}$. Then the symmetric solution to the quadratic inverse eigenvalue problem with positive definite mass matrix is given as follows:

$$M = Q \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} Q^T,\ C = Q \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} Q^T,\ K = Q \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} Q^T, \tag{3.5.21}$$

where

1. M is arbitrarily symmetric positive definite,

2. $C_{22} = C_{22}^T,\ K_{22} = K_{22}^T \in \mathbb{R}^{(n-k)\times(n-k)}$ are arbitrarily symmetric,

3. $C_{21} = C_{12}^T \in \mathbb{R}^{(n-k)\times k}$ is arbitrary,

4. $C_{11} = -(M_{11}S + S^T M_{11} + R^{-T} D R^{-1}) \in \mathbb{R}^{k\times k}$,

5. $K_{11} = S^T M_{11} S + R^{-T} D T R^{-1} \in \mathbb{R}^{k\times k}$,

6. $K_{21} = -(M_{21}S^2 + C_{21}S) \in \mathbb{R}^{(n-k)\times k}$,

with

$$D = diag\left(\begin{pmatrix} \sigma_1 & \eta_1 \\ \eta_1 & -\sigma_1 \end{pmatrix}, ..., \begin{pmatrix} \sigma_l & \eta_l \\ \eta_l & -\sigma_l \end{pmatrix}, \sigma_{2l+1}, ..., \sigma_k\right),$$

and σ's and η's being arbitrary real numbers. Further it is shown in [45], that matrices M_{11}, M_{21} and C_{21} can be chosen is such a way that resulting matrix K is positive definite.

3.5.1 Pencils with Banded Coefficient Matrices

In [66] Ram and Elhay, presented a method which constructs a tridiagonal symmetric quadratic pencil with $2n$ eigenvalues and the $2n-2$ of its $n-1$-dimensional leading principal subpencils prescribed. The mass matrix M is assumed to be an identity matrix, and

$$C = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & ... & 0 \\ \beta_1 & \alpha_2 & \beta_2 & ... & 0 \\ 0 & \beta_2 & \alpha_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \alpha_n \end{pmatrix}, \; K = \begin{pmatrix} \gamma_1 & \delta_1 & 0 & ... & 0 \\ \delta_1 & \gamma_2 & \delta_2 & ... & 0 \\ 0 & \delta_2 & \gamma_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \gamma_n \end{pmatrix} \tag{3.5.22}$$

Theorem 3.14 *Given two sets of distinct numbers $\{\lambda_j\}_{j=1}^{2n}$ and $\{\mu_j\}_{j=1}^{2n-2}$. Define a polynomial $p(\lambda) = \sum_{i=0}^{2n-2} c_i \lambda_i$, such that*

$$-p(\mu_j) = \prod_{i=1}^{2n} (\mu_j - \lambda_j), \; j = 1, ..., 2n-2,$$

and $\lambda = r$ is a double root of $p(\lambda)$, for some r, i.e.:

$$\frac{d}{d\lambda} p(r) = 0.$$

Then the last row of the resulting pencil can be identified as:

1. $$\alpha_n = \sum_{i=1}^{2n-2} \mu_i - \sum_{i=1}^{2n} \lambda_i,$$

2. $\beta_{n-1}^2 = c_{2n-2}$,

3. $\delta_{n-1} = -r\beta_{n-1}$,

4. $$\gamma_n = \frac{\prod_{i=1}^{2n}(r-\lambda_i)}{\prod_{i=1}^{2n-2}(r-\mu_i)} - r^2 - r\alpha_n.$$

Knowing the last row it is possible to determine the eigenvalues of the leading principal submatrices of dimension $n-2$ of $P(\lambda)$. Thus, the method prescribed in the theorem can be reapplied to the deescalated problem of dimension $n-1$. Authors presented a numerical algorithm based on these approach.

Another numerical method of solving inverse eigenvalue problem associated with a serially linked mass-spring system (2.2.6) was presented in [28]. Note, that in (2.2.6) damping and stiffness matrices are linear function of α's and β's, respectively, i.e., $C = C(\alpha),\ K = K(\beta)$. A Newton's iteration-based method was presented in [66] to find a zero of the function:

$$f(\alpha,\beta) = \begin{pmatrix} f_1(\alpha,\beta) \\ f_2(\alpha,\beta) \\ \vdots \\ f_{2n}(\alpha,\beta) \end{pmatrix},$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n)$, and

$$f_i(\alpha,\beta) = \det(\lambda_i^2 M + \lambda_i C(\alpha) + K(\beta)).$$

Recently, two more approaches were presented to solve tridiagonal inverse problem, [2, 13]. In his work [2] Bai considers the case of reconstructing a monic quadratic

tridiagonal pencil, when four eigenpairs $\{(\phi_i, \lambda_i)\}_{i=1}^4$ are given. In this case,

$$\Lambda^2\Phi + C\Lambda\Phi + K\Phi = 0,$$

is a system of $4n$ equations, when (Φ, Λ) is 4-matrix eigenpair, and can be rewritten as the following linear system:

$$Ay = g.$$

Note, that there are $4n$ equations and $4n-2$ unknowns, namely $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^{n-1}$, $\{\gamma_i\}_{i=1}^n$, and $\{\delta_i\}_{i=1}^{n-1}$. The results on existence and uniqueness of the solution to the problem are provided in [2].

In a similar manner, as in [2], the problem of reconstructing the mass-spring system (2.2.6) from two eigenpairs is considered in [13]. Note, there are $3n$ physical parameters, namely $\{m_i\}_{i=1}^n$, $\{\alpha_i\}_{i=1}^n$ and $\{\beta_i\}_{i=1}^n$. Thus, specifying more then three eigenpairs will make the problem over-determined. In [13] authors provide a condition under which the problem will have a solution, with positive coefficients m's, α's and β's. A numerical algorithm which constructs the masses $m_1, ..., m_n$, the damping constants $\alpha_1, ..., \alpha_n$, and the spring constants $\beta_1, ..., \beta_n$, all positive and $m_i \leq 1$, for the pencil $\lambda^2 M + \lambda C + K$ with the prescribed eigenpairs, where M,C, and K are of the structure specified in (2.2.6), or determines that such a system does not exists is also presented in [13].

3.5.2 An Application of Structured QIEP: Finite Element Model Updating

The model updating problem can be viewed as a QIEP. Since in a modal-based approach the matrices M, C, and K are assumed to be explicitly given, the problem here is to modify these matrices, rather than construct them from "the novo", using a given set of self-conjugate numbers and vectors, so that the updated model with the modified matrices contains the given numbers in its spectrum with the given

vectors as the corresponding eigenvectors, and the structure of the original model is preserved.

The problem routinely arises in vibration industries, including automobiles, air- and spacecraft. A properly updated model can be used with confidence for future design and analysis.

Because of its practical importance, the problem has been widely studied, both by mathematicians and practicing and academic engineers; as a result there now exists a voluminous body of work on the solutions.

For discussion of these methods, see the authoritive book [33], on the subject. For other recent methods, (see [10, 11, 52]).

In [10, 11, 52], the authors have solved the model updating problems with an additional constraint that the eigenvalues and eigenvectors of the original model which are not affected by updating, remain unchanged. This is known as the *no spill-over* in engineering literature. No spill-over constraint guarantees that "no spurious modes will be introduced into the frequency range of interest," which is a major concern for the engineers. However, there are some philosophical differences of opinion by the engineers about the "no spill-over."

Model updating problems are usually formulated as constrained optimization problems. Since damping is hard to estimate, very often in engineering practice, the model to be updated is considered to be an undamped model. In such cases, it is possible to give an explicit solution. See the book [33] for a discussion of such existing methods. However, in the case of damped model, the associated minimization problem is usually a nonlinear constrained minimization problem and numerical optimization techniques need to be used.

In the following section we will include a brief discussion on both unconstrained

and constrained minimization techniques. Note that a constrained minimization problem may be cast as unconstrained minimization problem. For details of the numerical optimization methods, see the book by Nocedal and Wright [60].

CHAPTER 4

Optimization

In this chapter, we very briefly review some of the concepts and numerical methods for optimization. Some of these concepts and techniques will be used later in the dissertation. For details, see the book by Nocedal and Wright [60].

Many of the inverse eigenvalue problems, are formulated as optimization problems. In several cases, it is possible to obtain a closed-form solution. For example, closed-form solution of some of the model updating problems can be obtained using Lagrange multiplier formalism [32, 33]. However, in most of the cases, the usage of numerical optimization methods is necessary [60]. In fact, an optimization algorithms, were used to obtain a solution to both of the problems considered in Chapter 5 of this dissertation.

4.1 Unconstrained Optimization

First, let's review a more simple case of an unconstrained optimization problem. The mathematical formulation is

$$\min_{x \in \mathbb{R}^n} f(x), \tag{4.1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. The following theorems give necessary and sufficient conditions for optimality of x^* (for proofs see [60]).

Theorem 4.1 (First-Order Necessary Conditions) *If x^* is a local minimizer and f is continuously differentiable in an open neighborhood of x^*, then $\nabla f(x^*) = 0$.*

Theorem 4.2 (Second-Order Necessary Conditions) *If x^* is a local minimizer and $\nabla^2 f$ is continuous in an open neighborhood of x^*, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite.*

Definition 4.1 *A point x^* is a strict local minimizer (also called a strong local minimizer) if there is a neighborhood $\mathcal{N}$ of x^* such that $f(x^*) < f(x)$ for all $x \in \mathcal{N} with x \neq x^*$.*

Theorem 4.3 (Second-Order Sufficient Conditions) *Suppose that $\nabla^2 f$ is continuous in an open neighborhood of x^*, and that $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite. Then x^* is a strict local minimizer of f.*

4.2 Constrained Optimization

The following portion of this section reviews the constraint optimization problem. We begin by reviewing the concept of Lagrange multipliers, which is a useful tool to treat constrained optimization problems. A general formulation of a constrained optimization problem is

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} f(x) \\ &\text{s.t. } c_i(x) = 0,\ i \in \mathcal{E} \\ &\quad\ c_i(x) \geq 0,\ i \in \mathcal{I}, \end{aligned} \tag{4.2.2}$$

where f and $c_i's$ are smooth real-valued functions on $\mathbb{R}^n$, and $\mathcal{E}$ and $\mathcal{I}$ are finite sets of indices. Function f is called the objective function and $c_i,\ i \in \mathcal{E}$ and $c_i,\ i \in \mathcal{I}$ are equality and inequality constraints correspondingly. The set of points, which satisfy the constants

$$\Omega = \{x \mid c_i(x) = 0,\ i \in \mathcal{E}, c_i \geq 0,\ i \in \mathcal{I}\},$$

is called the *feasibility set*, and the constrained optimization problem can be formulated as follows:

$$\min_{x \in \Omega} f(x).$$

Definition 4.2 *A point x^* is a global minimizer if $f(x^*) \leq f(x)$, for all $x \in \Omega$.*

Definition 4.3 *A point x^* is a local minimizer if $f(x^*) \leq f(x)$, for all $x \in \mathcal{N}(x^*)$, some neighborhood of x^*.*

We illustrate the concept by a simple example of finding an extreme of $f(x)$ with one equality constraint. In Figure 4.2, we can see a graph of the constraint equation $c_1(x) = 0$ and a family of curves $f(x) = s$, which cover a part of the space. Moving along the curve $c_1 = 0$, we cross curves from the family $f = s$, and parameter s changes monotonically. The solution to the problem is expected to be at the point, where the direction of s changes. From the Figure 4.2, we can see that it happens at the point where curve $f = s$ is "tangent" to the curve $c_1 = 0$. This means, that both of the curves $c_1 = 0$ and $f = s$ have common tangent line, thus, the normal vectors are collinear, i.e.,

$$\nabla f = \lambda_1 \nabla c_1.$$

By introducing the Lagrangian function:

$$L(x, \lambda_1) = f(x) - \lambda_1 c_1(x),$$

we can restate the necessary optimality condition as follows: At the solution x^*, there is a scalar λ_1^*, such that

$$\nabla_x L(x^*, \lambda_1^*) = \nabla f(x^*) - \lambda_1^* c_1(x^*) = 0.$$

The scalar λ_1 is called a *Lagrange multiplier*. The optimality condition for a problem

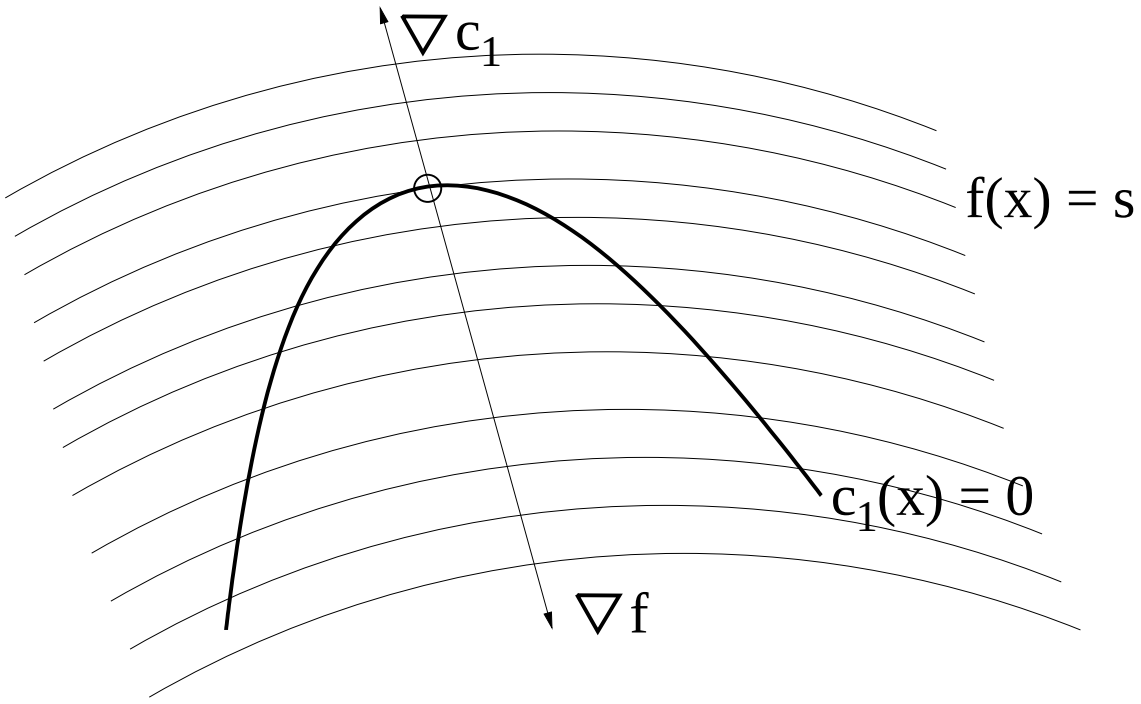

Figure 4.1: Constraint optimization problem.

with one inequality constraint can be formulated in a similar way, with a *complementary condition*:

$$\nabla_x L(x^*, \lambda^*) = 0, \quad \text{for some } \lambda_1^* \geq 0.$$

It also required that

$$\lambda_1^* c_1(x^*) = 0.$$

Thus, if $x^* \in \Omega$ is a local solution of (4.2.2), then there is a Lagrange multiplier vector λ^*, with components $\lambda_i^*,\ i \in \mathcal{E} \bigcup \mathcal{I}$, such that the following conditions are

satisfied:

$$\begin{aligned} \nabla_x L(x^*, \lambda^*) &= 0 \\ c_i(x^*) &= 0, \quad \text{for all } i \in \mathcal{E} \\ c_i(x^*) &\geq 0, \quad \text{for all } i \in \mathcal{I} \\ \lambda_i^* &\geq 0, \quad \text{for all } i \in \mathcal{I} \\ \lambda_i^* c_i(x^*) &= 0, \text{ for all } i \in \mathcal{E} \bigcup \mathcal{I}. \end{aligned} \tag{4.2.3}$$

These conditions 4.2.3 are usually called *Karush-Kuhn-Tucker* conditions or *KKT conditions* for short.

Definition 4.4 *At a feasible point x, the inequality constraint $i \in \mathcal{I}$ is said to be active if $c_i(x) = 0$ and inactive if the strict inequality $c_i(x) > 0$ is satisfied.*

Definition 4.5 *The active set $\mathcal{A}(x)$ at any feasible x is the union of the set $\mathcal{E}$ with the indices of the active inequality constraints; that is,*

$$\mathcal{A}(x) = \mathcal{E} \bigcup \{i \in \mathcal{I} \mid c_i(x) = 0\}.$$

Definition 4.6 *Given the point x^* and the active set $A(x^*)$ defined above, we say that the linear independence constraint qualification (LICQ) holds if the set of active constraint gradients $\{\nabla c_i(x^*),\ i \in \mathcal{A}(x^*)\}$ is linearly independent.*

Definition 4.7 *Given a feasible point x^*, a sequence $\{z_k\}_{k=0}^{\infty}$ with $z_k \in \mathbb{R}^n$ is a feasible sequence if the following properties hold:*

1. *$z_k \neq x^*$, for all k;*
2. *$\lim_{k \to \infty} z_k = x^*$*
3. *z_k is feasible for all sufficiently large k.*

Definition 4.8 *A limiting point of the following sequence:*

$$\frac{z_k - x}{||z_k - x||},$$

is called a limiting direction of the feasible sequence $\{z_k\}_{k=0}^{\infty}$.

The sequence $(z_k - x)/||z_k - x||$ lies on the surface of a compact set (unit sphere), and thus has at least one limiting point.

Lemma 4.1 *The following two statements hold.*

1. *If* $d \in \mathbb{R}^N$ *is a limiting direction of a feasible sequence, then*

$$d^T \nabla c_i^* = 0, \text{ for all } i \in \mathcal{E}, \ d^T \nabla c_i^* \geq 0, \text{ for all } i \in \mathcal{A}(x^*) \bigcap \mathcal{I}. \tag{4.2.4}$$

2. *If (4.2.4) holds with* $||d|| = 1$ *and the LICQ condition is satisfied, then* $d \in \mathbb{R}^n$ *is a limiting direction of some feasible sequence.*

For proof see [60].

The set of directions defined by (4.2.4) plays a central role in the optimality conditions, so for future reference we give this set a name and define it formally.

Definition 4.9 *Given a point* x^* *and the active constraint set* $A(x^*)$ *defined by (4.2.4), the set* F_1 *is defined by*

$$F_1 = \left\{ \alpha d \ | \alpha > 0, \quad \begin{array}{l} d^T \nabla c_i^* = 0, \text{ for all } i \in \mathcal{E} \\ d^T \nabla c_i^* \geq 0, \text{ for all } i \in \mathcal{A}(x^*) \bigcap \mathcal{I} \end{array} \right\}.$$

The cone F_1 is simply the set of all positive multiples of all limiting directions of all possible feasible sequences, see Lemma. (4.1).

Above, a necessary condition of optimality has been formulated. When KKT condition is satisfied a move along any vector w from F_1 either increases the first-order approximation to the objective function (that is, $w^T \nabla f(x^*) > 0$), or else

keeps this value the same (that is, $w^T \nabla f(x^*) = 0$). The second-order conditions concern the curvature of the Lagrangian function in the "undecided" directions, the directions $w \in F_1$ for which $w^T \nabla f(x^*) = 0$. Naturally, we need to assume that objective function and constraint functions have smooth second derivatives. Lets define subset of F_1, which contains "undecided" directions, by

$$F_2(\lambda^*) = \{w \in F_1 \mid w^T \nabla f(x^*) = 0, i \in \mathcal{A}(x^*) \bigcap \mathcal{I},\ \lambda_i^* > 0\}.$$

Here λ^* and x^* satisfy KKT conditions. Now second order sufficient condition can be formulated.

Theorem 4.4 *Suppose that for some feasible point $x^* \in \mathbb{R}^n$, there is a Lagrange multiplier vector λ^* such that the KKT conditions (4.2.3) are satisfied. Suppose also that*

$$w^T \nabla_{xx} L(x^*, \lambda^*) w > 0,\ \textit{for all } w \in F_2(\lambda^*), w \neq 0.$$

Then x^* is a strict local solution for (4.2.2).

4.3 Augmented Lagrangian Method

Augmented Lagrangian method for constrained optimization seeks the solution by replacing the original constrained problem by a sequence of unconstrained subproblems.

The problems considered in this dissertation are equality-constrained problems. In this section we will review augmented Lagrangian method applied to problems with equality constraints:

$$\min_x f(x), \text{subject to } c_i(x) = 0,\ i \in \mathcal{E}. \tag{4.3.5}$$

Definition 4.10 *Function*

$$Q(x; \rho) = f(x) + \rho \sum_{i \in \mathcal{E}} c_i^2(x)$$

is called the quadratic penalty function, and ρ is called the penalty parameter.

We can consider a sequence of values $\rho_k \to \infty$ and seek the approximate minimizer x_k of $Q(x;\rho_k)$ for each k. This type of methods is called quadratic penalty methods.

Theorem 4.5 *If $||\nabla Q(x;\rho_k)|| \leq \tau_k$ and $\tau_k \to 0$, while $\rho_k \to \infty$, then for all limit points x^* of $\{x_k\}$ at which $\nabla c_i(x^*)$ are linearly independent, we have that x^* is a KKT point of (4.3.5). If $\{x_k\}_{k\in\mathcal{K}} \to x^*$ is a convergence subsequence of x_k, then*

$$\lim_{k\in\mathcal{K}} \rho_k c_i(x_k) = -\lambda_i^*, \text{ for all } i \in \mathcal{E}, \tag{4.3.6}$$

where λ^* is a multiplier vector that satisfies the KKT conditions (4.2.3).

For the proof see [60]. Thus, the minimizers x_k of $Q(x;\rho_k)$ do not satisfy the feasibility condition; for large k we have

$$c_i(x_k) \approx -\frac{\lambda_i^*}{\rho_k}.$$

To get an optimizer which nearly satisfies the equality constraint, we introduce Lagrange multiplier term into the quadratic penalty function. It will lead us to augmented Lagrangian method. This "trick" reduces the possibility of ill-conditioning of the subproblems. The Lagrangian with an additional penalty term is called *augmented Lagrangian*:

$$L_\rho(x,\lambda) = f(x) - \sum_{i\in\mathcal{E}} \lambda_i c_i(x) + \rho_k \sum_{i\in\mathcal{E}} c_i^2(x). \tag{4.3.7}$$

The augmented Lagrangian method will be considered in more detail in Chapter 5.

CHAPTER 5

Model Updating

5.1 Mathematical Statement and Engineering and Computational Challenges

Finite element model updating problem concerns updating of an analytical symmetric finite element generated model using measured data from a real-life or an experimental structure. The updating needs to be done in such a way that the symmetry of the model is preserved and the updated model contains some of the desirable physical and structural properties of the original finite element model.

Mathematically, the problem can be formulated as follows: given a structured pencil (M_a, C_a, K_a) and a few of its associated eigenpairs $\{(\lambda_i, \phi_i)\}_{i=1}^k$ with $k << 2n$, assume that newly measured eigenpairs $\{(\mu_i, y_i)\}_{i=1}^k$ have been obtained. Update the pencil (M_a, C_a, K_a) to $(M_u, C_u, K_u) = (M_a + \Delta M, C_a + \Delta C, K_a + \Delta K)$ having the same structure, so that the subset $\{(\lambda_i, \phi_i)\}_{i=1}^k$ is replaced by $\{(\mu_i, y_i)\}_{i=1}^k$ as k eigenpairs of (M_u, C_u, K_u).

There now exists a large number of model updating methods. Most of this work prior to 1995 is contained in the book by Friswell and Mottershead [33]) and references therein. Some of the more recent results can be found in [10, 11, 25, 29, 32, 38, 42, 54].

Most existing methods concern updating of an undamped analytical model of the form:

$$M_a\ddot{x}(t) + K_a x(t) = 0,$$

where M_a, K_a are, respectively, the analytical mass and stiffness matrices. For the sake of convenience, we will denote this model simply by (M_a, K_a). Similar notations will be used for the updated model; that is, (M_u, K_u) will stand for the undamped updated model and (M_u, C_u, K_u) will stand for the *damped updated model*:

$$M_u\ddot{x} + C_u\dot{x} + K_u x = 0.$$

The analytical and updated eigenvalues and eigenvectors will also be denoted in a similar way.

A standard practice is to formulate the updating problem in an optimization setting such that the undamped updated model satisfies the following basic properties of the original model [33]:

(i) $M_u = M_u^T$

(ii) $K_u = K_u^T$ (*symmetry*)

(iii) $\Phi_u^T M_u \Phi_u = I$ (*orthogonality*)

(iv) $K_u\Phi_u = M_u\Phi_u\Lambda_u$ (*eigenvalue-eigenvector relation*).

Maintaining symmetry and reproduction of the measured data are the basic requirements for model updating. However, satisfaction of the orthogonality relation by the measured data is also of prime importance because the measured data, which comes from an experiment or a real-life structure, very often fails to satisfy the orthogonality constraint (iii).

Besides these basic requirements, there are also other engineering and computational challenges associated with the updating problem. These include: (i) dealing with incompleteness of the measured data, and (ii) complex measured data versus real analytical data, etc. Due to hardware limitations, the measured eigenvectors

are very often of much shorter length than the corresponding analytical eigenvectors. However, in order to use these measured data in updating process, these two sets of data must be of equal length. To remedy this situation, either the order of the analytical model is reduced *(model reduction)* or the measured eigenvectors are expanded *(model expansion)*.

For details, see the book by Friswell and Mottershead [33]. In this dissertation, we deal only with real representations of the data and assume that either modal expansion or model reduction has been performed to deal with the issue of the incomplete measured data.

5.2 Methods for Undamped Models

There are two types of updating procedures. The first type of methods, assuming the mass matrix as the reference matrix, update first the measured data so that it satisfies the mass-orthogonality constraint (iii). This is then followed by updating the stiffness matrix so as to satisfy the constraints (ii) and (iv).

Baruch and Bar Itzhack [4] suggested to update eigenvector matrix so that it satisfies the mass-orthogonality relation and the following mass-weighted norm is minimized:

$$\frac{1}{2}||M_a(\Phi_a - \Phi_u)|| \rightarrow \min$$
$$\Phi_u M_a \Phi_u = I.$$

An updated matrix K_u is a solution of the following minimization problem:

$$\frac{1}{2}||M_a^{-1/2}(K_a - K_u)M_a^{-1/2}|| \rightarrow \min$$
$$K_u = K_u^T.$$

The others update, either separately or simultaneously, the mass and stiffness

matrices, satisfying the constraints (i)-(iv) [9, 42, 75].

5.2.1 Undamped Model Updating with No Spurious Modes and with Incomplete Measured Data

There are two major engineering concerns with updating, one, as noted before, is the appearance of spurious modes in the frequency range of interest after updating, and the other, the issue of incomplete measured data. Due to hardware limitations, measured eigenvectors are almost always much shorter than their analytical counterparts. This difficulty is usually overcome by model expansion of the measured eigenvalues or reducing the order of the model.

There now exists a method which updates the stiffness matrix first satisfying the constraints (ii) and (iv) and then computes the missing entries of the measured modes in a computational setting such that computed data satisfies the mass orthogonality constraint [10]. The method proposed in [10] *has the additional important feature that the eigenvalues and eigenvectors which are not updated remain unchanged by the updating procedure.* This guarantees that "no spurious modes appear in the frequency range of interest."

To present the method proposed in [10], consider the following partitioning of eigenvalue and eigenvector matrices:

$$X = (X_1,\ X_2),\ \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix},$$

where $X_1 \in R^{n\times k},\ \Lambda_1 \in R^{k\times k}$. Let $\{\mu_1, ..., \mu_k\}$ and $\{y_1, ..., y_k\}$ be a set of k ($k << 2n$) eigenvalues and eigenvectors measured from an experimental structure. Let $Y = (y_1, ..., y_k)$ and $\Sigma = diag(\mu_1, ..., \mu_k)$. Let

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix},\ Y_1 \in R^{m\times m},\ Y_2 \in R^{(n-m)\times m}, \tag{5.2.1}$$

and it is assumed that only Y_1 is known. The question is, how we can construct Y_2

and how can we update the matrix K to K_u, such that

$$(Y,\ X_2),\ \begin{pmatrix} \Sigma & 0 \\ 0 & \Lambda_2 \end{pmatrix}$$

is the matrix eigenpair of the updated model? In other words, the updated model should reproduce measured eigenvectors and eigenvalues and the rest of the eigenstructure remains unchanged. The following orthogonality relations, proved in [10], play the central role in the proposed updating procedure.

Theorem 5.1 *Let $P(\lambda) = \lambda^2 M + K$ be a be a symmetric pencil with M positive definite and K semidefinite, with distinct nonzero eigenvalues and let (X, Λ) be the matrix eigenpair of this pencil. Then the matrices D_1 and D_2 defined by*

$$D_1 = X^T M X, D_2 = X^T K X$$

are diagonal and $D_2 = -D_1 \Lambda^2$.

(b) Furthermore, suppose that Λ_1 and Λ_2 do not have a common eigenvalue, then

$$X_1^T M X_2 = 0,\ \textit{and}\ X_1^T K X_2 = 0. \tag{5.2.2}$$

Theorem 5.2 (Eigenstructure Preserving Updating) *Assume that Λ_1 and Λ_2 do not have a common eigenvalue. Then, for every symmetric matrix Ψ , the updated pencil $P_u(\lambda) = \lambda^2 M + K_u$, where*

$$K_u = K - M X_1 \Psi X_1^T M \tag{5.2.3}$$

is such that

$$M X_2 \Lambda_2^2 + K_u X_2 = 0.$$

That is, the eigenvalues and eigenvectors of the original finite element model that are not to be affected by updating, remain unchanged.

Proof. Substituting the expression of K_u from (5.2.3) and using the orthogonality relation (5.2.2) above, we have

$$MX_2\Lambda_2^2 + K_uX_2 = MX_2\Lambda_2^2 + KX_2 - MX_1\Psi X_1^T MX_2 = 0$$

□

The question now is how to choose the matrix Ψ such that the measured eigenvalues and eigenvectors will be reproduced by the updated model. That is, for what choice of Ψ, the matrix K_u is such that

$$MY\Sigma^2 + K_uY = 0?$$

An algorithm description for choosing such a Ψ via solution of a Sylvester equation was given in [10]. The unmeasured part Y_2 of Y was also constructed in this algorithm satisfying the mass-orthogonality relation that Y^TMY is a diagonal matrix.

The following result shows how to choose the matrix Ψ such that it is symmetric and the measured eigenvalues and eigenvectors are embedded into the updated model.

Theorem 5.3 *Let matrix Y be mass-orthogonal; i.e., Y^TMY is a diagonal matrix. Let Ψ satisfy the Sylvester matrix equation:*

$$(Y^TMX_1)\Psi(Y^TMX_1)^T = Y^TMY\Sigma^2 + Y^TKY. \tag{5.2.4}$$

Then Ψ is symmetric and

$$MY\Sigma^2 + K_uY = 0, \textit{where } K_u \textit{ as in (5.2.3).} \tag{5.2.5}$$

Recently, Chu et al. [17] established a necessary and sufficient condition for existence of Ψ.

Theorem 5.4 *There exists matrix $\Psi \in R^{k\times k}$, such that*

$$MY\Sigma^2 + K_u Y = 0 \tag{5.2.6}$$

if and only if

$$Y = X_1 V D$$

for some orthogonal matrix $V \in R^{k\times k}$ and some nonsingular diagonal matrix $D \in R^{k\times k}$. Here $K_u = K - MX_1\Psi X_1^T M$.

The following is the algorithm described in [10] for computing Ψ and finding unmeasured part Y_2 of Y. First, find QR decomposition of MX_1:

$$MX_1 = (U_1\ U_2)\begin{pmatrix} Z \\ 0 \end{pmatrix}.$$

Now, substituting the expression for Y (5.2.1) into (5.2.6) and premultiplying by $(U_1\ U_2)^T$, we have

$$\begin{pmatrix} U_1^T \\ U_2^T \end{pmatrix}(M_1\ M_2)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}\Sigma^2 = (K_1\ K_2)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} Z \\ 0 \end{pmatrix}\Psi X_1^T MY,$$

where $M = (M_1\ M_2)$ and $K = (K_1\ K_2)$, $M_1, K_1 \in R^{n\times m}$ and $M_2, K_2 \in R^{n\times(n-m)}$. Compute now Y_2 by solving the Sylvester equation:

$$U_2^T M_2 Y_2 \Sigma^2 + U_2^T K_2 Y_2 = -U_2^T(K_1 Y_1 + M_2 Y_1 \Sigma_1^2). \tag{5.2.7}$$

Algorithm 1 Model Updating algorithm for an undamped model with guaranteed no spill-over

INPUT: $M = M^T > 0$, $K = K^T$, $\Sigma = diag(\mu_1, ..., \mu_k)$, Y_1
OUTPUT: Updated stiffness matrix K_u

1: Compute Y_2 by solving the equation (5.2.7) and form the matrix $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$
2: Mass-orthogonalize matrix Y by computing LDL decomposition of $Y^T M Y = LDL^T$. Update the matrix Y by $Y \leftarrow YL^{-T}$
3: Compute Ψ by solving the following algebraic system of equation:

$$(Y^T M X_1)\Psi(Y^T M X_1)^T = Y^T M Y \Sigma^2 + Y^T K Y,$$

4: Update the stiffness matrix:

$$K_u = K - M X_1 \Psi X_1^T M$$

5.3 Model Updating for Damped Models

Updating of a damped model is rarely considered in the engineering literature, the reason being, it is hard to estimate damping in practice. However, all the existing structures have damping. Earlier, several control-theoretic methods for updating were developed [56, 57, 76]. Unfortunately, the use of control destroys the symmetry. There are optimization-based techniques which are capable of preserving the symmetry. Friswell et al. [32] gave an explicit formula for updating a damped model, and a numerical algorithm based on the formula was developed by Kuo et al. [44].

In this section, we discuss two recent updating schemes for damped models: a symmetric, no spill-over eigenvalue embedding method by Carvalho et al. [12] and its generalization by Lancaster [50, 52] and a new two-stage optimization-based algorithm .

5.3.1 Eigenvalue Embedding Methods

Carvalho et al. [12] developed an updating scheme which preserves symmetry, and uses low-rank updating, and guarantees the *no spill-over* with mathematical proof. Unfortunately, in their work, only updating of measured eigenvalues were considered. The scheme did not take into account the updating of the measured eigenvectors. The proposed updating scheme, however, preserves the symmetry of the original model.

We illustrate the scheme by means of the following theorem, where a real isolated eigenvalue is updated. But the process is more general and is capable of updating both real and complex eigenvalues.

Theorem 5.5 *Let (λ, x) be a real isolated eigenpair of $P(\lambda) = \lambda^2 M + \lambda D + K$ with $x^T K x = 1$. Let λ be reassigned to μ. Assume that $1 - \lambda\mu\theta \neq 0$ and $1 - \lambda^2\theta \neq 0$, where $\theta = y^T M y$ and $\epsilon = (\lambda - \mu)/(1 - \lambda\mu\theta)$. Then the following updated quadratic matrix pencil:*

$$P_u(\lambda) = \lambda^2 M_u + \lambda C_u + K_u$$

with

$$M_u = M - \epsilon\lambda M x x^T M$$
$$C_u = C + \epsilon(M x x^T K + K y x^T M)$$
$$K_u = K - \frac{\epsilon}{\lambda} K x x^T K,$$

is such that

i. The eigenvalues of $P_u(\lambda)$ are the same as those of $P(\lambda)$ except that λ has been replaced by μ (assignment of real eigenvalues).

ii. x_1 is also an eigenvector of $P_u(\lambda)$ corresponding to the embedded eigenvalue μ.

iii. If (λ_2, x_2) is an eigenpair of $P(\lambda)$, where $\lambda_2 \neq \lambda$, then (λ_2, y_2) is also an eigenpair of $P_u(\lambda)$ (no spill-over).

A generalization of the result is obtained by Lancaster in [50, 52]. In his work, Lancaster considers Hermitian matrices (instead of real symmetric). The method presented by Lancaster is also capable of updating measured eigenvectors. The moments (3.2.5) defined previously are used to solve the problem. Recall that the coefficients of $P(\lambda)$ can be recursively determined in terms of the moments:

$$\Gamma_j = X\Lambda^j P X^*, j = 1, 2, 3$$

$$M = \Gamma_1^{-1},\ C = -M\Gamma_2 M,\ K = -M\Gamma_3 M + C\Gamma_1 C,$$

where P is a permutation matrix, defined in (3.3.16). Now we consider not only the splitting of the spectral data but also the splitting of the matrix P:

$$X = (X_1,\ X_2),\ \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix},\ P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

As before, Λ_1 is to be modified to Σ and Λ_2 is its unknown complement and stays unchanged after updating. The eigenvectors X_1 associated with Λ_1 are to be replaced by Y. The spectral decomposition of M gives

$$M^{-1} = X\Lambda P X^* = S_1 + S_2,$$

where $S_1 = X_1\Lambda_1 P_1 X_1^*,\ S_2 = X_2\Lambda_2 P_2 X_2^*$. This approach admits only changes from real to real eigenvalues and complex conjugate pairs to complex conjugate pairs. Thus, matrix P remains unchanged. The updated matrix M_u, is now defined by

$$\boxed{M_u^{-1} = M^{-1} + (\overline{S}_1 - S_1)}$$

where $\overline{S}_1 = Y\Sigma P_1 Y^*$. For updating the damping matrix, the matrices T_1 and T_2 are introduced as:

$$T_1 = X_1\Lambda_1^2 P_1 X_1^*,\ T_2 = T_1 = X_2\Lambda_2^2 P_2 X_2^*,$$

then $\Gamma_2 = T_1 + T_2$. Now

$$C = -M(T_1 + T_2)M,\ D_u = -M_u(\overline{T}_1 + T_2)M,$$

where $\overline{T}_1 = Y\Sigma^2 P_1 Y^*$. Elimination of T_2 leads to

$$\boxed{C_u = M_u(M^{-1}CM^{-1} - (\overline{T}_1 - T_1))M_u.}$$

For updating the matrix K, define $U_1,\ U_2$ by

$$U_1 = X_1\Lambda_1^3 P_1 X_1^*,\ U_2 = X_2\Lambda_2^3 P_2 X_2^*,$$

so that $\Gamma_3 = U_1 + U_2$, and then the updated matrix $\overline{U}_1 = Y\Sigma^3 P_1 X_1^*$. Then

$$K = -M(U_1 + U_2)M + CM^{-1}C,\ K_u = -M_u(\overline{U}_1 + U_2)M_u + C_u M_u^{-1} C_u.$$

Elimination of U_2 leads to

$$\boxed{K_u = -M_u(M^{-1}(CM^{-1}C - K)M^{-1} + (\overline{U}_1 - U_1)M_U + C_u M_u^{-1} C.}$$

Note that the formulas for (M_u, C_u, K_u) do not require the knowledge of the (X_2, Λ_2).

5.4 Quadratic Model Updating with Measured Data Satisfying Orthogonally Relations: A New Method

As stated before, most existing updating methods for an undamped model implicitly or explicitly use orthogonality constraint. Satisfaction of an orthogonality constraint, of course, is essential for measured data to be acceptable for updating. However, the role orthogonality constraint plays in updating procedure itself has not been systematically investigated.

In this section, we prove, both for undamped and damped models, that satisfaction of the orthogonality relation in both cases is necessary and sufficient for

preserving the symmetry in the updated model. Using this new result, we then propose a two-stage optimization process for updating a damped model.

In Stage I, we update the measured eigenvectors so that they satisfy a quadratic orthogonality relation (2.3.9). In Stage II, the updated measured eigenvectors from Stage I are used to update the stiffness matrix so that it remains symmetric after updating and the measured eigenvalues and eigenvectors are reproduced by the updated model. *Thus, our method generalizes methods for undamped models of the first type to a damped model.* The results of numerical experiments on some case studies are presented to show the accuracy of the proposed method.

The problems in both stages are nonlinear optimization problems. The Stage I problem is a nonconvex minimization problem with equality constraint. This is a difficult optimization problem to solve. An augmented Lagrangian method is proposed to deal with this problem. Some convergence properties of this method are discussed.

The Stage II problem is a convex quadratic problem. This is a rather nice optimization problem to deal with and there are several excellent numerical methods for such problems in the literature (see [60]). Necessary optimization background also has been reviewed in the Chapter 4.

Implementations in optimization settings of Stage I and Stage II require that the appropriate gradient formulas must be computed in terms of the known quantities only, which are, in our case, just a few measured eigenvalues and eigenvectors and the corresponding sets from the analytical model. Such gradient formulas have been mathematically derived.

5.4.1 Existence of Symmetric Solution of the Model Updating Problem

In this section, we establish mathematical results to demonstrate the fact that the updated measured data must satisfy orthogonality condition for the existence of a symmetric solution to the Stage II. We consider both cases: undamped and damped models.

5.4.2 Linear Case (*Undamped Model*)

Consider the symmetric undamped model

$$M\ddot{x}(t) + Kx(t) = 0.$$

Let (X, T) be the real-form representation of the complex eigenpair (Φ, Λ) of the associated generalized eigenvalue problem.

Theorem 5.6 *Given $M = M^T > 0$, $T \in \mathbb{R}^{k\times k}$, a block-diagonal matrix of the form (2.5.16), $X \in \mathbb{R}^{n\times k}$ of the form (2.5.15) with full rank, there exists a symmetric nonzero matrix K such that $KX = MXT$ if and only if $X^TMX = B$, where B is a block-diagonal matrix of the form:*

$$B = diag(B_1, ..., B_l, B_{2l+1}, ..., B_k),\ B_j = \begin{cases} \begin{pmatrix} a_j & b_j \\ b_j & -a_j \end{pmatrix},\ j = 1, ..., l \\ b_j,\ j = 2l+1, ..., k \end{cases} \tag{5.4.8}$$

Proof. ($\Leftarrow$) Sufficiency. Since X has full rank, the matrix equation $KX = MXT$ has a nonzero solution. To prove that there exists a symmetric solution K to the equation, we consider an extension of (X, T) in the form $X_{ext} = [X\ \hat{X}] \in \mathbb{R}^{n\times n}$, $T_{ext} = diag(T, \hat{T}) \in \mathbb{R}^{n\times n}$ such that $X_{ext}^T M X_{ext} = B_{ext} = diag(B, \hat{B})$, where $\hat{B}$ is a block-diagonal matrix, X_{ext} is of full rank, and $\hat{T}$ is a block-diagonal matrix. Now define

$$K = X_{ext}^{-T} B_{ext} T_{ext} X_{ext}^{-1}.$$

Then, obviously, $KX = MXT$; moreover, since $B_{ext}T_{ext}$ is a symmetric matrix, K is also symmetric. Different choices of $\hat{X}$ and $\hat{T}$ will produce different symmetric solutions to the above equation.

($\Rightarrow$) Necessity. Since $M = M^T > 0$ and $K = K^T$, there exists a matrix Φ, such that $\Phi^T M \Phi = D$, where D is a diagonal matrix (see [18]).

Setting $X = \Phi S^{-1}$, where S is defined as in (2.5.14), we have, $X^T M X = S^{-T}\Phi^T M \Phi S^{-1} = S^{-T} D S^{-1} = B$. Thus, a 2×2 block of B is of the following form:

$$\sqrt{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-T} \begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix} \sqrt{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

□

5.4.3 Quadratic Case (*Damped Model*)

In this section, we prove an analogous result for the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$. In this case there are $2n$ eigenvalues and eigenvectors.

The real-form representation of the matrix eigenpair (Φ, Λ) of $P(\lambda)$ is denoted by (X, T).

The pair (X, T) satisfies the relation (2.5.17), which can be written as

$$\begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} X \\ XT \end{pmatrix} = \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} X \\ XT \end{pmatrix} T. \tag{5.4.9}$$

This shows that the matrix

$$\begin{pmatrix} X \\ XT \end{pmatrix}$$

is the real-form representation of the eigenvector matrix of the $2n \times 2n$ linear pencil

$$\left(\begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix}, \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \right).$$

Since M, C, K are symmetric, by Theorem 5.6, therefore, we have

$$\begin{pmatrix} X \\ XT \end{pmatrix}^T \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} X \\ XT \end{pmatrix} = B(X,T), \tag{5.4.10}$$

where $B = diag(B_1, ..., B_l, B_{2l+1}, ..., B_k)$ is a block-diagonal matrix with blocks defined as in (5.4.8). Note the relation (5.4.10) is equivalent to the orthogonality relation (2.3.9) for the complex eigenpair.

To solve the inverse problem, i.e., to find a symmetric K which will satisfy the eigenvalue-eigenvector relation (2.5.17) for given $M = M^T > 0, C = C^T$, and the rank condition (X,T) satisfying (2.5.18), we find an extension of the matrices X and T, $X_{ext} = [X\ \hat{X}] \in \mathbb{R}^{n\times n}$, $T_{ext} = diag(T, \hat{T}) \in \mathbb{R}^{n\times n}$, such that

$$\begin{pmatrix} X_{ext} \\ X_{ext}T_{ext} \end{pmatrix} \text{is of full rank and}$$

$$\begin{pmatrix} X_{ext} \\ X_{ext}T_{ext} \end{pmatrix}^T \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} X_{ext} \\ X_{ext}T_{ext} \end{pmatrix} = [B(X,T), \hat{B}] = B_{ext}(X_{ext}, T_{ext}).$$

Here $\hat{T}, \hat{B}$ are real block-diagonal matrices. Then we can take K as the solution to the following linear system:

$$\begin{pmatrix} X_{ext} \\ X_{ext}T_{ext} \end{pmatrix}^T \begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} X_{ext} \\ X_{ext}T_{ext} \end{pmatrix} = B_{ext}T_{ext}$$

i.e., $K = X_{ext}^{-T}(T_{ext}^T X_{ext}^T M X_{ext} T_{ext} - B_{ext}T_{ext})X_{ext}^{-1}$. This is a real symmetric matrix (note, that $B_{ext}T_{ext}$ is a symmetric matrix).

The above discussion leads to the following theorem.

Theorem 5.7 *Given $M = M^T > 0 \in \mathbb{R}^{n\times n}$, $C = C^T \in \mathbb{R}^{n\times n}$, and $T \in \mathbb{R}^{k\times k}$, $X \in \mathbb{R}^{n\times k}$, $k < n$ matrices of the form (2.5.16), (2.5.15), respectively. Let (X,T) satisfy condition (2.5.18). Then there is a real symmetric matrix K such that $MXT^2 + CXT + KX = 0$ if and only if*

$$\begin{pmatrix} X \\ XT \end{pmatrix}^T \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} X \\ XT \end{pmatrix} = B(X,T), \tag{5.4.11}$$

where B is some block-diagonal matrix with blocks of the form (5.4.8).

Corollary 5.1 *Assume that T is a real-form representation of an eigenvalue matrix of the symmetric positive definite pencil (M, C, K) and all the diagonal blocks of T are distinct, then B is of form (5.4.8) if and only if $BT = T^T B$.*

Proof. Consider the matrix D_3 from (2.3.9), and note that $B = S^{-T} D_3 S^{-1}$. Then, $BT = S^{-T} D_3 S^{-1} S \Lambda S^{-1} = S^{-T} D_3 \Lambda S^{-1}$ and $T^T B = S^{-T} \Lambda S^T S^{-T} D_3 S^{-1} = S^{-T} \Lambda D_3 S^{-1}$. Now, $B = S^{-T} D_3 S^{-1}$ is of the form (5.4.8) if and only if $D = S^T B S$ is a diagonal matrix, which is equivalent, to

$$\Lambda D_3 = D_3 \Lambda.$$

This implies that $BT = T^T B$. □

5.4.4 A Two-Stage Model Updating Scheme

In this section, we introduce our two-stage model updating scheme for FEMU. Throughout, we assume that M_a, C_a, K_a, $X_M = \Phi_M S^{-1}$, and $\mathcal{T}_M = S \Lambda_M S^{-1}$ are given, $M_a > 0$, $K_a = K_a^T$, $C_a = C_a^T$.

Stage I: In this stage, the real-form representation of the measured eigenvector matrix X_M is updated so that it becomes as close as possible to the analytical data in the sense that a weighted distance between them is minimized. Furthermore, an orthogonality constraint stated in Corollary 5.1 is enforced. Mathematically, the problem may be stated as follows:

$$(\mathcal{P}) \qquad \begin{aligned} \min \quad & \frac{1}{2} ||W_1^{-1/2} (X - X_M) W_1^{-1/2})||_F^2 \\ \text{s.t.} \quad & H(X) = 0 \\ & X \in \mathbb{R}^{n \times k}, \end{aligned}$$

where

$$H(X) = B(X, T_M)T_M - T_M^T B(X, T_M), \tag{5.4.12}$$

with $B(X, T_M)$ given by (5.4.10) and W_1 is some positive-definite weighting matrix. A solution to the problem will be denoted by X_u.

Stage II : Let X_u be a solution from Stage I. In this stage, we would like to update the stiffness matrix K so that it becomes as close as possible to K_a in the sense that a weighted distance between K and K_a is minimized. In addition, constraints on symmetry for K and eigenvalue-eigenvector relation of (2.5.17) are enforced. Mathematically, this amounts to solving the following minimization problem:

$$\begin{aligned} (\mathcal{Q}) \qquad \min \quad & \frac{1}{2}||W_2^{-1/2}(K - K_a)W_2^{-1/2}||_F^2 \\ \text{s.t.} \quad & M_a X_u T_M^2 + C_a X_u T_M + K X_u = 0 \\ & K^T = K \quad \text{(symmetry)} \\ & K \in \mathbb{R}^{n \times n}, \end{aligned}$$

where $W_2 > 0$ is a positive-definite weighting matrix. The solution to the problem will be denoted by K_u.

Problem $(\mathcal{Q})$ is a convex quadratic programming problem with a unique solution. There exists an analytical expression [32] and a computational algorithm [44] based on numerical linear algebra techniques. Since $(\mathcal{Q})$ is a simple convex quadratic programming problem, we can also solve it numerically by existing optimization techniques.

Remarks: It is also possible to update both the stiffness and damping matrices satisfying the orthogonality relation of Stage I. This will require reformulation of

the problem. Such reformulation is currently being investigated.

5.5 A Solution Method and Its Convergence Properties

We now focus on how to solve $(\mathcal{P})$. As noted before Problem $(\mathcal{Q})$ is a convex quadratic programming problem for which there exist excellent numerical methods. However, in our numerical experiment, we use the same method developed for $(\mathcal{P})$ to solve $(\mathcal{Q})$.

To simplify the presentation, set $W = I$, and

$$f(X) = \frac{1}{2}||X - X_M||_F^2.$$

Then the Lagrangian function for $(\mathcal{P})$ is:

$$L(X, Y) = f(X) + \langle Y, H(X)\rangle,$$

where $Y \in \mathbb{R}^{k\times k}$. Some remarks about our Lagrange function L are in order. By definition, $H(X)^T = -H(X)$. Hence the system $H(X) = 0$ defines $k(k-1)/2$ constraints.

The necessary optimality conditions for $(\mathcal{P})$ can now be stated as follows: Find a pair (X_*, Y_*) such that

$$\nabla_X L(X_*, Y_*) = 0, \quad H(X_*) = 0. \tag{5.5.13}$$

Problem $(\mathcal{P})$ has a convex quadratic objective function, and polynomial equality constraints. So it is a polynomial programming problem. But the feasible region defined by the polynomial equality constraints are nonconvex in general. Hence we are dealing with a nonconvex minimization problem with equality constraints.

Generally speaking, if X_* is an optimal solution for $(\mathcal{P})$, and the constraint system satisfies certain regularity condition at X_*, then there is a $Y_* \in \mathbb{R}^{k\times k}$ such that (X_*, Y_*) satisfies (5.5.13). Elements of Y_* are usually called Lagrangian multipliers.

Optimization techniques for constrained problems such as $(\mathcal{P})$ have been well developed in the past fifty years. There are many efficient methods, including augmented Lagrangian methods, to solve a nonlinear programming problem with equality constraints. The first augmented Lagrangian method was independently proposed by Hestenes [41] and Powell [62] by adding a quadratic penalty term to its Lagrangian function $L(X, Y)$. Because of its attractive features, such as ease to implement, it has emerged as an important method for handling constrained optimization problems. The literature on augmented Lagrangian methods is vast. We refer the reader to [6, 67] for a thorough treatment on this class of methods and its convergence theory. Following Hestenes and Powell, we propose an augmented Lagrangian method to solve $(\mathcal{P})$. To this end, we introduce the following parameterized family of the augmented Lagrangian functions:

$$L_\rho(X, Y) = L(X, Y) + \frac{\rho}{2}||H(X)||_F^2, \tag{5.5.14}$$

where ρ is a positive constant.

We now discuss an issue associated with our proposed augmented Lagrangian method.

The existence of a global minimizer in Step 3: For $\mathcal{P}(i)$, the following proposition guarantees that there is a solution.

Proposition 5.1 *Let $\rho > 0$ and $Y \in \mathbb{R}^{k\times k}$. Then $argmin_X L_\rho(X, Y)$ is non-empty.*

Proof. We will prove that $L_\rho(\cdot, Y)$ is level-bounded [67, Definition 1.8]; that is, for each real μ, the set $\{X \mid L_\rho(X, Y) \leq \mu\}$ is bounded. Once this is done, the non-emptiness of $argmin_X L_\rho(X, Y)$ follows from [67, Theorem 1.9].

Let $\{X_i\}$ be a sequence such that $||X_i||_F \to \infty$ as $i \to \infty$. Then

$$L_\rho(X_i, Y) \to \infty \text{ as } i \to \infty$$

Algorithm 2 The augmented Lagrangian method for $(\mathcal{P})$

INPUT: $X_0, Y_0, \rho_0 > 0, 0 < \beta < 1$, and $\epsilon > 0$

OUTPUT: Solution to $(\mathcal{P})$

1: **for** i = 0,1,... **do**

2: Stop if $||\nabla_X L(X_i, Y_i)|| \leq \epsilon$, and $||H(X_i)|| \leq \epsilon$

3: Solve the unconstrained optimization subproblem:

$$(\mathcal{P}(i)) \quad \min_X \; L_{\rho_i}(X, Y_i)$$

$$\text{with the stopping criteria, } ||\nabla_X L_{\rho_i}(X, Y_i)||_F < \beta^i$$

Let X_{i+1} be the solution of $(\mathcal{P}(\text{i}))$

4: Update the multiplier matrix:

$$Y_{i+1} = Y_i + \rho_i H(X_{i+1})$$

Then choose $\rho_{i+1} > \rho_i$

5: **end for**

since, by Cauchy-Schwartz inequality $\langle Y, H(X_i)\rangle \geq -||Y||_F||H(X_i)||_F$,

$$\langle Y, H(X_i)\rangle + \rho/2||H(X_i)||_F^2 \geq (\rho/2||H(X_i)||_F - ||Y||_F)||H(X_i)||_F$$

and $f(X_i) \to \infty$ as $i \to \infty$. If $\{X : \; L_\rho(X, Y) \leq \mu\}$ were not a bounded set for some μ, then there would exist a sequence $\{X_i\}$ sequence such that $||X_i||_F \to \infty$ as $i \to \infty$. This would imply that

$$\mu \geq f(X_i) + \langle Y, H(X_i)\rangle + \rho/2||H(X_i)||_F^2 \to +\infty$$

as $i \to \infty$ by the above argument. The contradiction proves the level-boundedness of $L_\rho(\cdot, Y)$. □

2. *The convergence of the proposed method:* As we have already pointed out before, $(\mathcal{P})$ is a nonconvex programming problem. It is well known in optimization literature that finding a global minimizer for a nonconvex programming problem is a very challenging task. A practical way for solving a sequence of nonconvex

programming problems in Step 3 is to find a sequence of critical points instead. The following theorem, which is embedded somewhere in the general convergence theory in [6, 67], ensures that such a sequence of critical points will have a convergent subsequence.

To make our presentation self-contained and for the reader's convenience, we include a proof of this theorem in the next section after computable gradient formulas are derived. Let us set

$$F(X) = [H_{12}(X), \cdots, H_{1k}(X), H_{23}(X), \cdots, H_{2k}(X), \cdots, H_{(k-1)k}(X)]^T,$$

and

$$\hat{Y} = [Y_{12}, \cdots, Y_{1k}, Y_{23}, \cdots, Y_{2k}, \cdots, Y_{(k-1)k}]^T. \tag{5.5.15}$$

Then $F : \mathbb{R}^{k\times k} \rightarrow \mathbb{R}^{(k-1)k/2}$, and it is easy to see that $F(X) = 0$ if and only if $H(X) = 0$. Also, simple calculations show that

$$< Y, H(X) >= 2 < \hat{Y}, F(X) >$$

and

$$||H(X)||_F^2 = 2||F(X)||_F^2.$$

Our Lagrangian functions then become

$$L(X, Y) = L(X, \hat{Y}) = f(X) + 2 < \hat{Y}, F(X) >$$

and

$$L_\rho(X, Y) = L_\rho(X, \hat{Y}) = L(X, \hat{Y}) + \rho||F(X)||_F^2.$$

Theorem 5.8 *Suppose* $||\nabla_X L_{\rho_j}(X_j, \hat{Y}_j)||_F < \beta^j,\ 0 < \beta < 1$ *for* $j = 0, 1, 2, \ldots,$ *with* $\{||\hat{Y}_j||_F\}$ *is bounded, and* $\rho_j < \rho_{j+1}$ *and* $\rho_j \rightarrow \infty$ *as* $j \rightarrow \infty$. *If there is*

a convergent subsequence $\{X_{j_i}\}$ *of* $\{X_j\}$ *with* $X_{j_i} \to X_*$ *such that* $\nabla F(X_*)$ *maps* $\mathbb{R}^{n\times k}$ *onto* $\mathbb{R}^{[(k-1)\times k/2]\times[n\times k]}$, *then there is a* $\hat{Y}_*$ *such that*

$$\nabla_X L(X_*, \hat{Y}_*) = 0, \quad F(X_*) = 0. \tag{5.5.16}$$

Remarks: (a) In view of the relationships between F and H, $\hat{Y}$ and Y, it is easy to see that $\nabla_X L(X_*, \hat{Y}_*) = 0$ and $F(X_*) = 0$ if and only if there is some Y_* (by (5.5.15)) such that the pair (X_*, Y_*) satisfies (5.5.13). (b) It is also clear from (5.5.15) that $\{||Y_j||_F\}$ is bounded if and only if $\{||\hat{Y}_j||_F\}$ is bounded. The following simple example shows that if the conditions listed in Theorem 5.8 are not satisfied, Lagrangian multipliers may not exist.

Example 1 *Consider* $min_{x\in R}\ \ x$ *subject to* $x^2 = 0$. *Then it is trivial to see that* $x = 0$ *is an optimal solution. However, there is no Lagrangian multiplier for the minimization problem.*

5.5.1 Computation of the Gradient Formulas

To implement the proposed method effectively, we need to have gradient formulas that can be computed in terms of the given quantities. In this section, we will show that gradients of functions associated with problems $(\mathcal{P})$ and $(\mathcal{Q})$ can be computed in terms of their associated matrices. This is particularly important since our numerical experiments are conducted in MATLAB environment.

The basic idea for deriving gradient formulas comes from operator theory on adjoint operators as has been used in [14].

It is easy and elementary to see that

$$\nabla f(X) = X - X_M.$$

Let

$$\begin{aligned} h(X) &= 1/2||H(X)||_F^2 = 1/2\langle H(X), H(X)\rangle \\ g_Y(X) &= \langle Y, H(X)\rangle. \end{aligned}$$

Then

$$\nabla h(X) = 2(M_a X G(X) {T_M}^T + (C_a X + M_a X T_M) G(X)), \tag{5.5.17}$$

where $G(X) = H(X){T_M}^T - T_M H(X)$ and $H(X)$ is given by (5.4.12):

$$\nabla g_Y(X) = 2(M_a X (Y T_M^T - T_M Y) T_M^T + (C_a X + M_a X T_M)(Y T_M^T - T_M Y)). \tag{5.5.18}$$

The details of the derivation of formula (5.5.17) are given in the following subsection.

With the above gradient formulas, the gradient formulas for L and L_ρ with respect to X can be written as:

$$\begin{aligned} \nabla_X L(X,Y) &= \nabla f(X) + \nabla g_Y(X), \\ \nabla_X L_\rho(X,Y) &= \nabla_X L(X,Y) + \rho \nabla h(X). \end{aligned}$$

5.5.2 Gradient Formulas for Problem $\mathcal{Q}$

Gradient functions in Problem $(\mathcal{Q})$ are much simpler, and can be written down as follows. For $\bar{h}(K) = 1/2||K - K_a||_F^2$, we have

$$\nabla \bar{h}(K) = K - K_a.$$

For $\bar{f}(K) = 1/2||M_a X_u T_M^2 + C_a X_u T_M + K X_u||_F^2$, we have

$$\nabla \bar{f}(K) = (M_a X_u T_M^2 + C_a X_u T_M + K X_u) X_u^T.$$

We will now use the above gradient formulas to obtain the necessary optimality conditions for Stage I.

Necessary optimality conditions in matrix form for (P):

Find X and Y such that

$$\begin{aligned} \nabla f(X) + \nabla g_Y(X) &= 0, \\ H(X) &= 0, \end{aligned}$$

where $H(X)$is given by (5.4.12), and $\nabla g_Y(X)$ is given by (5.5.18). The above necessary optimality conditions expressed in terms of given matrices are significant. It not only opens up the possibility of solving $(\mathcal{P})$ by solving the above systems of equations (such as by Newton's method for nonlinear equations), but it also forms the basis for sensitivity analysis when the problem data undergoes small changes.

We conclude this section by including a proof of Theorem 5.8. We begin with a well-known lemma on the invertibility of a matrix (see e.g., [18, p. 319]).

Lemma 5.1 *Let p, q be positive integers such that $p \geq q$. Suppose that A is an $p \times q$ matrix with rank q. Then $[A^T A]^{-1}$ exists.*

Proof. The proof is by contradiction. Let u be a nonzero $q \times 1$ vector such that $A^T Au = 0$. Then $(Au)^T(Au) = 0$. It follows that $Au = 0$. The rank condition on A implies that $u = 0$, which is a contradiction. This proves $A^T A$ is invertible. □

Proof of Theorem 5.8. Without any loss of generality, suppose that $X_j \to X_*$ as $j \to \infty$. For any fixed X, since $\nabla F(X)$ is a $[(k-1) \times k/2] \times [n \times k]$ matrix, $\nabla F(X_*)$ has rank $(k-1) \times k/2$, and $\nabla F(\cdot)$ is continuous. We may further assume that $\nabla F(X_j)$ has rank $(k-1) \times k/2$ for all j. Set $A_j = \nabla F(X_j)^T$ and $A_* = \nabla F(X_*)^T$. By Lemma 5.1, $A_j^T A_j$ is invertible. The continuity of $\nabla F(\cdot)$ implies that $[A_j^T A_j]^{-1} \to [A_*^T A_*]^{-1}$. By

$$\nabla_X L_{\rho_j}(X_j, \hat{Y}_j) = \nabla f(X_j) + 2\nabla F(X_j)^T[\hat{Y}_j + \rho_j F(X_j)], \tag{5.5.19}$$

and multiplying both sides of (5.5.19) by $\nabla F(X_j) = A_j^T$, we have

$$2(\hat{Y}_j + \rho_j F(X_j)) = [A_j^T A_j]^{-1} A_j^T [\nabla_X L_{\rho_j}(X_j, \hat{Y}_j) - \nabla f(X_j)]$$

Since $||\nabla_X L_{\rho_j}(X_j, \hat{Y}_j)||_F \to 0$ and $[A_j^T A_j]^{-1} \to [A_*^T A_*]^{-1}$ as $j \to \infty$, we get

$$2(\hat{Y}_j + \rho_j F(X_j)) \to -[A_*^T A_*]^{-1} A_*^T \nabla f(X_*) \quad \text{as} \quad j \to \infty.$$

Set $2\hat{Y}_* = -[A_*^T A_*]^{-1} A_*^T \nabla f(X_*)$. By taking the limit on both sides (5.5.19), we have that $\nabla_X L(X_*, \hat{Y}_*) = 0$. Since $2(\hat{Y}_j + \rho_j F(X_j)) \to 2\hat{Y}_*$, and $\{||\hat{Y}_j||_F\}$ is bounded, the sequence $\{\rho_j F(X_j)\}$ is bounded. By $\rho_j \to \infty$ as $j \to \infty$, we conclude that

$$\lim_{j \to \infty} F(X_j) = F(X_*) = 0.$$

So the pair $(X_*, \hat{Y}_*)$ satisfies (5.5.16). This completes the proof.

5.5.3 Case Studies

In this section, we present the results of our numerical experiments on

- A spring-mass system of 10 degree of freedom (DoF) [33]
- A vibrating beam.

The data for our experiments are set up as follows:

- The matrices M_a, C_a are kept fixed.
- To simulate the measured data (X_M, T_M), we add a random noise with some Gaussian distribution to the eigendata of the analytical model.
- The weighting matrix was taken as $W = I$.

We used MATLAB with double arithmetics to run numerical experiments. As a solver of an unconstrained optimization problem, MATLAB optimization toolbox routine fminunc, which implements BFGS quasi-Newton method was used.

5.5.4 A Mass-Spring System of 10 DoF

Consider the example of a mass-spring system of 10 DoF, as depicted in Figure 5.5.4. In this example all rigid bodies have a mass of 1 kg, and all springs have stiffness 1 kN/m. The analytical model is given by

$$M_a = I$$

$$C_a = \begin{pmatrix} 0.48 & -8.3 & & & & & & & & \\ -8.38 & 8.38 & -1.02 & & & & & & & \\ & -1.0 & 1.02 & -7.28 & & & & & & \\ & & -7.28 & 7.28 & -4.40 & & & & & \\ & & & -4.40 & 4.40 & -9.97 & & & & \\ & & & & -9.97 & 9.97 & -5.62 & & & \\ & & & & & -5.62 & 5.62 & -4.65 & & \\ & & & & & & -4.65 & 4.65 & -4.19 & \\ & & & & & & & -4.19 & 4.19 & -2.11 \\ & & & & & & & & -2.11 & 2.11 \end{pmatrix}$$

$$K_a = \begin{pmatrix} 2000 & -1000 & & & & & & & & \\ -1000 & 3000 & -1000 & & -1000 & & & & & \\ & -1000 & 2000 & -1000 & & & & & & \\ & & -1000 & 3000 & -1000 & & & -1000 & & \\ & -1000 & & -1000 & 3000 & -1000 & & & & \\ & & & & -1000 & 2000 & -1000 & & & \\ & & & & & -1000 & 2000 & -1000 & & \\ & & & -1000 & & & -1000 & 3000 & -1000 & \\ & & & & & & & -1000 & 2000 & -1000 \\ & & & & & & & & -1000 & 2000 \end{pmatrix}$$

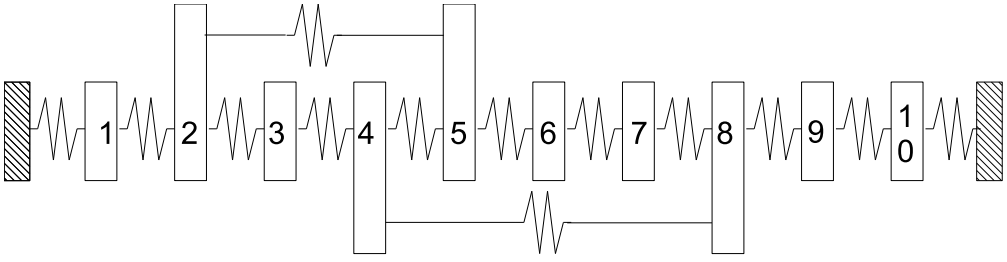

Figure 5.1: Mass - spring system with 10 DoF.

The measured data for our experiment was simulated by reducing stiffness of the spring between masses 2 and 5 to 600 N/m and adding Gaussian noise with $\sigma = 2\%$.

The analytical eigenvalue and eigenvector matrices are

$$T_a = \begin{pmatrix} \text{-6.23} & \text{71.1} & 0 & 0 \\ \text{-71.1} & \text{-6.23} & 0 & 0 \\ 0 & 0 & \text{-3.67} & \text{65.9} \\ 0 & 0 & \text{-65.9} & \text{-3.67} \end{pmatrix}, X_a = \begin{pmatrix} 0.142 & 0.001 & \text{-0.161} & \text{-0.001} \\ \text{-0.438} & 0.020 & 0.372 & \text{-0.050} \\ 0.288 & 0.065 & \text{-0.056} & 0.031 \\ \text{-0.502} & \text{-0.206} & \text{-0.191} & 0.087 \\ 0.479 & 0.148 & \text{-0.296} & \text{-0.034} \\ \text{-0.136} & \text{-0.011} & 0.263 & 0.091 \\ \text{-0.066} & 0.011 & \text{-0.346} & \text{-0.145} \\ 0.339 & 0.003 & 0.599 & 0.063 \\ \text{-0.122} & \text{-0.007} & \text{-0.296} & \text{-0.093} \\ 0.040 & 0.010 & 0.115 & 0.075 \end{pmatrix}$$

The measured eigenvalue and eigenvector matrices are:

$$T_M = \begin{pmatrix} \text{-6.16} & \text{69.8} & 0 & 0 \\ \text{-69.8} & \text{-6.16} & 0 & 0 \\ 0 & 0 & \text{-4.7} & \text{64.9} \\ 0 & 0 & \text{-64.9} & \text{-4.7} \end{pmatrix}, X_M = \begin{pmatrix} 0.102 & 0.026 & \text{-0.172} & \text{-0.023} \\ \text{-0.283} & \text{-0.061} & 0.401 & 0.023 \\ 0.282 & 0.115 & \text{-0.195} & \text{-0.005} \\ \text{-0.579} & \text{-0.240} & 0.074 & 0.068 \\ 0.341 & 0.242 & \text{-0.354} & \text{-0.202} \\ \text{-0.067} & \text{-0.054} & 0.286 & 0.242 \\ \text{-0.168} & 0.036 & \text{-0.249} & \text{-0.340} \\ 0.508 & \text{-0.042} & 0.362 & 0.382 \\ \text{-0.207} & 0.009 & \text{-0.183} & \text{-0.246} \\ 0.077 & 0.011 & 0.060 & 0.130 \end{pmatrix}$$

Results of Stage I

Orthogonality condition was not satisfied with the measured eigenvector matrix as shown by the residual:

$$||H(X_M)||_F = 17.7.$$

However, with preprocessing of the measured data using Stage I, we obtained

$$X_u = \begin{pmatrix} 0.086 & 0.128 & 0.139 & 0.021 \\ -0.270 & -0.226 & -0.324 & 0.055 \\ 0.266 & 0.176 & 0.233 & -0.131 \\ -0.437 & -0.239 & -0.222 & 0.264 \\ 0.227 & 0.387 & 0.343 & 0.041 \\ 0.031 & -0.261 & -0.218 & -0.232 \\ -0.147 & 0.205 & 0.183 & 0.312 \\ 0.395 & -0.239 & -0.226 & -0.544 \\ -0.145 & 0.142 & 0.149 & 0.244 \\ 0.051 & -0.061 & -0.064 & -0.096 \end{pmatrix}$$

The orthogonality condition is now satisfied with this updated eigenvector matrix as shown by the following residual:

$$||H(X_u)||_F = 1.97 \times 10^{-8}.$$

Results of Stage II

With the updated eigenvector matrix X_u from Stage I:

- The updated matrix K_u was symmetric, as shown by the residual norm:

$$||K_u - K_u^T||_F = 9.293 \times 10^{-9}.$$

- The measured eigenvalues and corrected measured eigenvectors were reproduced accurately by the updated model, as shown by the following residual:

$$||R(K_u)||_F = 1.78143 \times 10^{-6},$$

where $R(K) = M_a X_u T_M^2 + C_a X_u T_M + K X_u$.

Note: It is clear from the Figure 5.5.4 that the largest changes correspond to degrees of freedom 2 and 5. The changes corresponding to the other degrees of freedom are reasonably small.

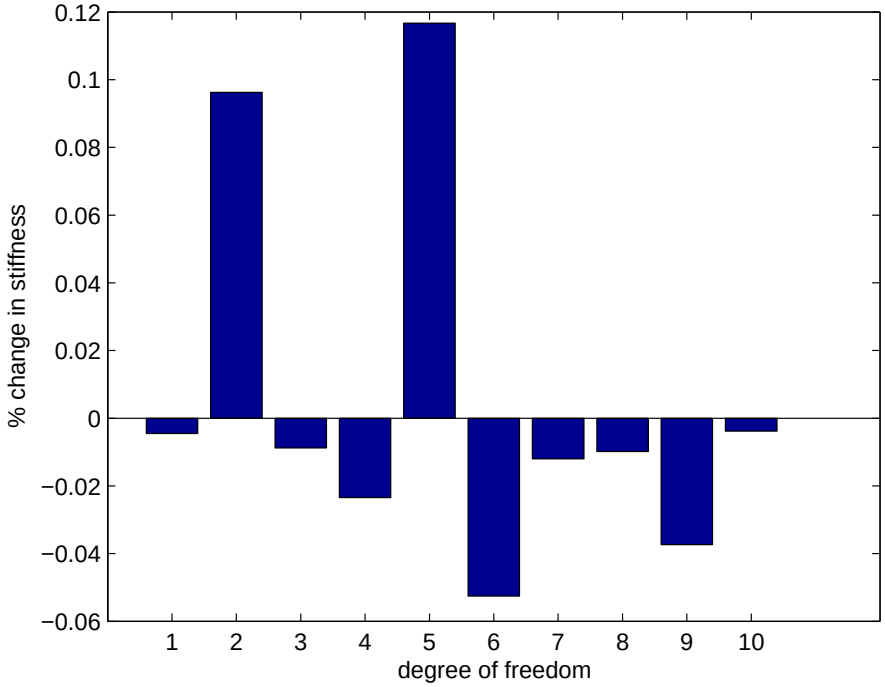

Figure 5.2: The percentage change in the diagonal elements of the stiffness matrix for mass-spring system.

5.5.5 Vibrating Beam

Consider a discrete spring-mass model of a vibrating beam [35], which consists of $n+2$ masses $\{m_i\}_{i=-1}^{n}$, linked by massless rigid rods of length $\{l_i\}_{i=0}^{n}$ which are themselves connected by n rotational springs of stiffness $\{k_i\}_{i=1}^{n}$. This model corresponds to a finite difference approximation of a beam with distributed parameters. The vibration of the beam with clamped left hand end and with no force applied at the free end is governed by:

$$M\ddot{x} + Kx = 0,$$

where

$$K = EL^{-1}E\hat{K}E^{T}L^{-1}E^{T}$$

$\hat{K} = diag(k_1, \ldots, k_n)$, $L = diag(l_1, \ldots, l_n)$, $M = diag(m_1, \ldots, m_n)$,

$$E = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 & -1 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

The simulated beam has 16 rods of length 1/16 m each, and all masses are 0.1 kg.

Results of Stage I

The measured data was obtained from the analytical data in the same way as the previous example. To simulate the measured data, the coefficients k_3, k_5, k_9 were reduced by 40%, 50%, 30%, respectively and Gaussian noises with $\sigma = 2\%$ were added. The simulated measured eigenvector matrix became:

$$X_M = \begin{pmatrix} -1.2528 \times 10^{-5} & \text{-0.00029345} & \text{-0.0028605} \\ 7.9745 \times 10^{-5} & 0.0015719 & 0.012768 \\ \text{-0.00039017} & \text{-0.0064281} & \text{-0.043326} \\ 0.0015785 & 0.021449 & 0.11279 \\ \text{-0.0054324} & \text{-0.058119} & \text{-0.23438} \\ 0.016097 & 0.13210 & 0.38359 \\ \text{-0.041476} & \text{-0.24960} & \text{-0.47393} \\ 0.091780 & 0.39290 & 0.39510 \\ \text{-0.17778} & \text{-0.49152} & \text{-0.10852} \\ 0.29897 & 0.45559 & \text{-0.24409} \\ \text{-0.44078} & \text{-0.23769} & 0.37962 \\ 0.53922 & \text{-0.10088} & \text{-0.16311} \\ \text{-0.52197} & 0.36497 & \text{-0.22086} \\ 0.34568 & \text{-0.34533} & 0.33064 \\ \text{-0.10101} & 0.11494 & \text{-0.12827} \end{pmatrix}$$

Without application of Stage I, the matrix X_M did not satisfy the orthogonality constraint, as shown by the residual:

$$||H(X_M)||_F = 3.668 \times 10^7.$$

Application of Stage I yielded:

$$X_u = \begin{pmatrix} -1.6363 \times 10^{-5} & \text{-0.00029433} & \text{-0.0028606} \\ 9.6687 \times 10^{-5} & 0.0015758 & 0.012768 \\ \text{-0.00044692} & \text{-0.0064414} & \text{-0.043328} \\ 0.0017231 & 0.021483 & 0.11280 \\ \text{-0.0057238} & \text{-0.058189} & \text{-0.23441} \\ 0.016549 & 0.13220 & 0.38365 \\ \text{-0.041977} & \text{-0.24972} & \text{-0.47406} \\ 0.092072 & 0.39297 & 0.39535 \\ \text{-0.17760} & \text{-0.49144} & \text{-0.10893} \\ 0.29832 & 0.45532 & \text{-0.24353} \\ \text{-0.44009} & \text{-0.23728} & 0.37893 \\ 0.53906 & \text{-0.10129} & \text{-0.16238} \\ \text{-0.52253} & 0.36525 & \text{-0.22148} \\ 0.34638 & \text{-0.34546} & 0.33102 \\ \text{-0.10127} & 0.11497 & \text{-0.12838} \end{pmatrix}$$

The updated matrix X_u did satisfy the orthogonality constraint, as shown by the residual:

$$||H(X_u)||_F = 9.09143 \times 10^{-6}.$$

Results of Stage II

With the updated eigenvector matrix X_u from Stage I:

- The updated matrix K_u was symmetric, as shown by the residual norm:

$$||K_u - K_u^T||_F = 9.897 \times 10^{-8}.$$

- The measured eigenvalues and the corrected measured eigenvectors were reproduced accurately by the updated model, as shown by the following residual:

$$||R(K_u)||_F = 5.795 \times 10^{-5}.$$

CHAPTER 6

Affine Parametric Quadratic Inverse Eigenvalue Problem

6.1 Introduction

In this chapter, we consider QIEP for the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$, in which matrices C and K are defined as members of the affine families

$$C = C(\alpha) = C_0 + \sum_{i=1}^{n} \alpha_i C_i, \;\; K = K(\alpha) = K_0 + \sum_{i=1}^{n} \beta_i K_i, \tag{6.1.1}$$

where $\alpha, \beta \in \mathbb{R}^n$, and the matrices C_i and K_i are real symmetric $n \times n$ matrices which comprise an affine family. For the sake of convenience we will denote this pencil as $(M, C(\alpha), K(\beta))$, and the eigenvalues of the pencil will be denoted by

$$\{\lambda_1(\alpha, \beta), ..., \lambda_{2n}(\alpha, \beta)\}.$$

The set of eigenvalues is closed under conjugation. The *affine quadratic inverse eigenvalue problem* to be considered here is stated as follows:

Problem 1. **Affine Quadratic Inverse Eigenvalue Problem (AQIEP)**. Given a set with distinct entries $\{\mu_1, ..., \mu_{2n}\}$, closed under complex conjugation, find $(\alpha, \beta) \in \mathbb{R}^{2n}$, such that $\{\mu_1, ..., \mu_{2n}\}$ are eigenvalues of the pencil $(M, C(\alpha), K(\beta))$. We will call the set $\{\mu_1, ..., \mu_{2n}\}$, the **target set of eigenvalues**. Matrix of target eigenvalues will be denoted by Σ

$$\Sigma = diag(\mu_1, ...\mu_{2n}).$$

As an example of how an affine model arises in practice, consider the following mass-spring system with damping.

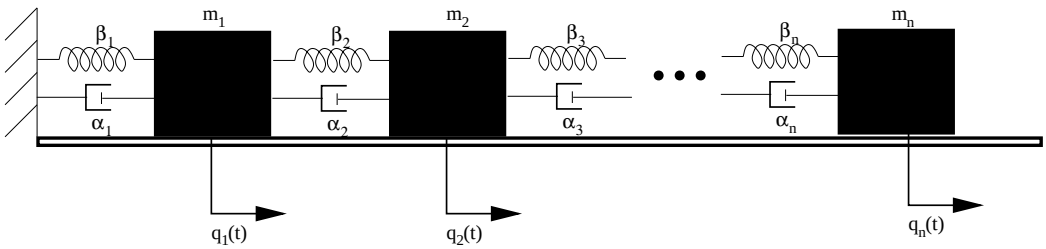

Figure 6.1: Serially linked mass-spring system.

The natural frequencies of the system are determined by the eigenvalues of the matrix pencil $(M, C(\alpha), K(\beta))$, where

$$M = \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix},$$

$$C = \begin{pmatrix} \alpha_1 + \alpha_2 & -\alpha_2 & & \\ -\alpha_2 & \alpha_2 + \alpha_3 & -\alpha_3 & \\ & \ddots & \ddots & \\ & & -\alpha_n & \alpha_n \end{pmatrix},$$

$$K = \begin{pmatrix} \beta_1 + \beta_2 & -\beta_2 & & \\ -\beta_2 & \beta_2 + \beta_3 & -\beta_3 & \\ & \ddots & \ddots & \\ & & -\beta_n & \beta_n \end{pmatrix}$$

Here, we see, that the matrices C and K are the members of the affine families of the form (6.1.1), where

$$K_0 = C_0 = 0,\ K_1 = C_1 = e_1 e_1^T$$

$$K_i = C_i = (e_{i-1} - e_i)(e_{i-1} - e_i)^T,\ i = 2, 3, ..., n.$$

Our purpose here is to first develop a Newton's method for solving Problem 1. It can be done by finding a zero of the function

$$f(\alpha, \beta) = \begin{pmatrix} \lambda_{\sigma(1)}(\alpha, \beta) - \mu_1 \\ \vdots \\ \lambda_{\sigma(2n)}(\alpha, \beta) - \mu_{2n} \end{pmatrix}, \tag{6.1.2}$$

where $\{\mu_1, ..., \mu_{2n}\}$ are the target eigenvalues and $\{\lambda_1(\alpha, \beta), ..., \lambda_{2n}(\alpha, \beta)\}$ are the eigenvalues of the pencil $(M, C(\alpha), K(\beta))$ in some order, and σ chosen in such a way that $\sum(\lambda_{\sigma(i)}(\alpha, \beta) - \mu_i)^2$ is minimum over all permutations σ of $1, 2, ..., 2n$. Newton's method for a standard inverse eigenvalue problem was thoroughly studied in the well-known paper by Friedland, Nocedal and Overton [31]. In [31] the authors considered the formulation and local analysis of four various quadratically convergent iterative methods, which are related to Newton's method, for solving the symmetric standard inverse eigenvalue problem. One of the approaches considered in the paper is to find a zero of the function f, which is the difference between set of prescribed eigenvalues and those calculated during iteration.

Note, that the formulation (6.1.2) requires the pairing of eigenvalues $\lambda(\alpha_i, \beta_i)$ with μ_i at each iteration. To avoid pairing problem, Elhay and Ram [28] proposed Newton's method based on minimizing the function $g(\alpha, \beta) = (g_1(\alpha, \beta), ..., g_{2n}(\alpha, \beta))$, where

$$g_i(\alpha, \beta) = det(\mu_i^2 M + \mu_i C + K). \tag{6.1.3}$$

However, it was noted in [31], that the approach of finding a zero of the function f defined in (6.1.2) is always to be preferred over the approach of finding zero of the function g defined above in (6.1.3).

The problem of finding permutation σ, needed to calculate value of f in (6.1.2), is known as the *matching problem*. Namely, given the two sets of numbers $a = \{a_1, ..., a_k\}$ and $b = \{b_1, ..., b_k\}$ we say that set $\{a_{\sigma(1)}, ..., a_{\sigma(k)}\}$ matches set $\{b_1, ..., b_k\}$ if σ is the permutation which minimizes

$$\sum_{j=1}^{k} (a_{\sigma(j)} - b_j)^2.$$

Permutation σ in (6.1.2) is the permutation which minimizes

$$\sum_{j=1}^{2n}(\lambda_{\sigma(j)}(\alpha,\beta)-\mu_j)^2 \tag{6.1.4}$$

among all possible permutations of the list of eigenvalues $\lambda_i(\alpha,\beta)$:

$$\sigma \in \arg\min_{\sigma(1),\ldots,\sigma(n!)} \sum_{j=1}^{2n}(\lambda_{\sigma(j)}(\alpha,\beta)-\mu_j)^2. \tag{6.1.5}$$

In other words, $\{\lambda_{\sigma_1}(\alpha,\beta),...\lambda_{\sigma_{2n}}(\alpha,\beta)\}$ is the closest match for the target eigenvalues $\{\mu_1,...,\mu_{2n}\}$.

In the context of ordering the real sets, we recall the well-known result [39].

Theorem 6.1 ([39]) *Given two sets of real numbers $a = \{a_1,...,a_k\}$ and $b = \{b_1,...,b_k\}$, the expression*

$$\sum_{j=1}^{k}(a_j-b_j)^2$$

has the minimum value, when a and b are both monotonically increasing or both monotonically decreasing, i.e., $a_1 \le ... \le a_k$ and $b_1 \le ... \le b_k$ or $a_1 \ge ... \ge a_k$ and $b_1 \ge ... \ge b_k$.

Matching problem for two sets of complex numbers can be solved in $O(n^3)$ time using the so-called Hungarian method [43].

Implementation of Newton's method for zero finding problem requires computation of Jacobian. In Section 6.2, we develop a formula for Jacobian in terms of λ_i's and the matrices M, C_k, and K_k, $k = 1, 2, ..., 2n$.

It is well known that Newton's method is locally convergent. In order to obtain a globally convergent method for our problem, we propose a hybrid method combining Newton's method with a globally convergent *alternating projections* method. The latter is developed with a view to obtaining a good initial guess for Newton's method.

A numerically efficient approach for generating projection operators, needed for alternating projections method, is also developed in this chapter. Finally, an efficient implementation of the proposed alternating projections method by computing the necessary eigenvectors using inverse iteration is suggested.

The chapter is organized as follows: In Section 6.2, we describe a Newton's method for solving the affine quadratic inverse eigenvalue problem and calculate the derivatives necessary for the Newton's method. We also show that the Newton's method is quadratically convergent. In Section 6.4 we develop the alternating projections method for solving affine QIEP and derive projection operators necessary for implementation of this method. As an improvement to alternating projections method we suggested using inverse iterations for computing eigenvalues. This also reduces the computational requirement, not only because we can calculate eigenvectors cheaply but also because the matching problem is solved automatically. Some global convergence results for the method are also provided in this chapter. Solution to the matrix nearness problem, which arises while calculating one of the projections operators, is presented in Section 6.3. We also show how a good approximation of the solution to the nearness problem can be calculated in a numerically efficient way. In Section 6.6 we present results of our numerical experiments which illustrate the accuracy of our hybrid method.

6.2 Newton's Method

Theorem 6.2 *Let (α^*, β^*) be a solution to Problem 1. Then there exists a neighborhood of (α^*, β^*) which contains no singular points of the spectra. These are the points where the pencil has multiple eigenvalues.*

For details see [5, 36, 69].

Corollary 6.1 *There is a neighborhood of* (α^*, β^*) *where* $\lambda_i(\alpha, \beta)$ *are distinct and are differentiable functions.*

Thus, the function f defined above in (6.1.2) is a differentiable function in a neighborhood of a solution.

The system (6.1.2) can be solved by applying Newton's method to it. To develop a gradient formula for the function $f(\alpha, \beta)$, let's first rewrite the quadratic eigenvalue problem $M\Lambda(\alpha,\beta)^2 X(\alpha,\beta) + C(\alpha)\Lambda(\alpha,\beta)X(\alpha,\beta) + K(\beta)X(\alpha,\beta) = 0$ as the symmetric generalized eigenvalue problem:

$$\begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} X \\ X\Lambda \end{pmatrix} = \begin{pmatrix} C & M \\ M & O \end{pmatrix} \begin{pmatrix} X \\ X\Lambda \end{pmatrix} \Lambda.$$

To make expressions more readable we have omitted arguments. We know, that eigenvector matrix of the above generalized eigenvalue problem satisfies the following orthogonality relations:

$$\begin{pmatrix} X \\ X\Lambda \end{pmatrix}^T \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} X \\ X\Lambda \end{pmatrix} = D$$

and

$$\begin{pmatrix} X \\ X\Lambda \end{pmatrix}^T \begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} X \\ X\Lambda \end{pmatrix} = D\Lambda,$$

where D is some diagonal matrix. Assuming that $D_{ii} = x_i^T(2\lambda_i M + C)x_i \neq 0,\ i = 1, ..., 2n$ and then taking

$$z_i = \frac{x_i}{\sqrt{x_i^T(2\lambda_i M + C)x_i}}$$

we can scale the eigenvectors so that $D = I$. Gradient formulas now can be obtained by differentiating the relations:

$$z_i^T(2\lambda_i M + C)z_i = 1$$

$$z_i^T(\lambda_i^2 M - K)z_i = \lambda_i$$

$$\dot{z}_i^T(2\lambda_i M + C)z_i + z_i^T(2\dot{\lambda}_i M + \dot{C})z_i + z_i^T(2\lambda_i M + C)\dot{z}_i = 0 \tag{6.2.6}$$

$$\dot{z}_i^T(\lambda_i^2 M - K)z_i + z_i^T(2\dot{\lambda}_i \lambda M - \dot{K})z_i + z_i^T(\lambda_i^2 M - K)\dot{z}_i = 0 \tag{6.2.7}$$

Multiplying (6.2.6) by λ and subtracting it from (6.2.7) we obtain

$$\dot{\lambda}_i = -z_i^T(\lambda \dot{C} + \dot{K})z_i. \tag{6.2.8}$$

Since C and K do not depend on β and α respectively, we can write

$$\frac{\partial \lambda_i}{\partial \alpha_k} = -\lambda_i z_i^T C_k z_i, \ \frac{\partial \lambda_i}{\partial \beta_k} = -z_i^T K_k z_i.$$

If we don't scale eigenvalues, then the derivatives are given by

$$\frac{\partial \lambda_i}{\partial \alpha_k} = -\frac{\lambda_i x_i^T C_k x_i}{x_i^T(2\lambda_i M + C)x_i}, \ \frac{\partial \lambda_i}{\partial \beta_k} = -\frac{x_i^T K_k x_i}{x_i^T(2\lambda_i M + C)x_i},$$

Thus the Jacobian of f is

$$J_{ik} = \begin{cases} \frac{\partial \lambda_i}{\partial \alpha_k} = -\frac{\lambda_i x_i^T C_k x_i}{x_i^T(2\lambda_i M + C)x_i}, \ k = 1, ..., n \\ \frac{\partial \lambda_i}{\partial \beta_{k-n}} = -\frac{x_i^T K_{k-n} x_i}{x_i^T(2\lambda_i M + C)x_i}, \ k = n+1, ..., 2n \end{cases}, \tag{6.2.9}$$

and a step of Newton's is defined as follows:

$$J(\alpha^i, \beta^i) \begin{pmatrix} \alpha^{i+1} - \alpha^i \\ \beta^{i+1} - \beta^i \end{pmatrix} = -f(\alpha^i, \beta^i). \tag{6.2.10}$$

Algorithm 3 Newton's Method

INPUT: λ^*, (α_0, β_0), tolerance - ϵ
OUTPUT: Solution to (6.1.2) - (α^*, β^*)

1: **for** i = 0,1,... **do**
2: Find eigenvalues and eigenvectors of $(M, C(\alpha_i), K(\beta_i))$.
3: Solve the minimization combinatorics problem (6.1.4) and compute $f(\alpha_i, \beta_i)$
4: Stop if $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon$
5: Calculate $J(\alpha^i, \beta^i)$ as given in (6.2.9) and find $(\alpha^{i+1}, \beta^{i+1})$ by solving (6.2.10)
6: **end for**

The advantage of using Newton's method is that it is quadratically convergent in a neighborhood of a solution in which $J(\alpha, \beta)$ is nonsingular. Thus, provided

an initial point is close enough to a solution, the Newton's iterations will rapidly converge (in practice 3-5 iterations). We will discuss in Section 6.4 how to obtain a point, which is close to a solution, using a globally convergent alternating projections method. The fact that a combinatorial minimization problem (linear assignment problem) must be solved at each iteration is not very restrictive. As it was mentioned above, this problem can be solved in $O(n^3)$ time using the so-called Hungarian method [43]. Moreover, in practice, we observed that as we approach to a solution, there is no need to compute σ since it gets stabilized, i.e., does not change from iteration to iteration.

6.3 Matrix Nearness Problem

Before we continue with alternating projections method, we need to consider matrix nearness problem, which arises while computing one of the projection operators needed for the algorithm proposed in the next section. The problem formulated as follows: given matrices $M, \hat{C}, \hat{K}$, find matrices C, K which are as close as possible to $\hat{C}, \hat{K}$ and so that the given matrix Σ is the eigenvalue matrix of the pencil $(M, \hat{C}, \hat{K})$. In other words,

$$\begin{aligned} &||C - \hat{C}||^2 + ||K - \hat{K}||^2 \rightarrow \min \\ &\text{s.t. } MX\Sigma^2 + \hat{C}\Sigma + \hat{K}X = 0 \\ &\hat{C}, \hat{K} \in R^{n\times n},\ X \in C^{n\times 2n},\ ||x_i|| = 1. \end{aligned} \tag{6.3.11}$$

Matrices C and K can be explicitly written in terms of X and Λ. Thus X is the only variable in (6.3.11). Recall the results shown in Section 2.1.5; given a nonsingular matrix M, a diagonal matrix Σ and some matrix X, so that $col(X, X\Sigma)$ is a nonsingular matrix, the relation

$$MX\Sigma^2 + \hat{C}X\Sigma + \hat{K}X = 0$$

holds if and only if

$$\hat{C} = -MX\Sigma^2 V_2 \tag{6.3.12}$$

$$\hat{K} = -MX\Sigma^2 V_1, \tag{6.3.13}$$

where V is the $2n \times 2n$ matrix

$$V = \begin{pmatrix} X \\ X\Sigma \end{pmatrix}^{-1} = (V_1, V_2),\ V_1, V_2 \in \mathbb{C}^{2n\times n}. \tag{6.3.14}$$

Thus, the optimization problem (6.3.11) can be rewritten as an unconstrained optimization problem:

$$F(X) = ||C - MX\Sigma^2 V_2||^2 + ||K - MX\Sigma^2 V_1||^2 \to \min. \tag{6.3.15}$$

The problem (6.3.15) can be solved by solving an equivalent nearness problem, formulated for the companion form linearization of the pencil, assuming that M is nonsingular:

$$\min_X ||A - Z\Sigma Z^{-1}||^2 \tag{6.3.16}$$

$$Z = col(X; X\Sigma),\ X \in \mathbb{C}^{n\times n}, \tag{6.3.17}$$

where

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix}.$$

Consider problem (6.3.16) disregarding the form of matrix Z (6.3.17). Let $H = Z\Sigma Z^{-1}$; then

$$F(Z) = ||A - Z\Sigma Z^{-1}||^2 =< A - H, A - H >=$$

$$< A, A > + < H, H > -2 < A, H >= ||A||^2 + tr(H^*H) - 2tr(A^*H).$$

The derivative of the function F can be easily found:

$$\frac{\partial}{\partial Z_{ij}} tr(A^*H) = tr\left(A^* \frac{\partial}{\partial Z_{ij}} \left(Z\Sigma Z^{-1}\right)\right) = tr\left(A^* \left(\frac{\partial Z}{\partial Z_{ij}} \Sigma Z^{-1} - Z\Sigma Z^{-1} \frac{\partial Z}{\partial Z_{ij}} Z^{-1}\right)\right).$$

Now

$$dtr(A^*H) = \nabla_Z tr(A^*H) = AZ^{-*}\Sigma - Z^{-*}\Sigma Z^* AZ^{-*}.$$

Note

$$dtr(A^*H) = dtr(H^*A) = tr(A^*dH) = tr(dH^*A).$$

Thus,

$$dtr(H^*H) = tr(dH^*H) + tr(H^*dH) = 2(HZ^{-*}\Sigma - Z^{-*}\Sigma Z^* HZ^{-*}).$$

We now can write the matrix of derivatives with respect to $Z_{i,j}$'s:

$$F_Z = 2(HZ^{-*}\Sigma - Z^{-*}\Sigma Z^* HZ^{-*}) - 2(AZ^{-*}\Sigma - Z^{-*}\Sigma Z^* AZ^{-*}) = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}. \quad (6.3.18)$$

$$(F_Z)_{i,j} = \frac{\partial}{\partial Z_{ij}} F(Z).$$

Note that if A is a normal matrix and we reduce the search space to unitary matrices Z, i.e., consider the solution of the form $X = Z\Sigma Z^*,\ ZZ^* = I$, then The gradient of F is the tangent vector is given by

$$\nabla_Z F = F_Z - ZF_Z^T Z = AZ\Sigma - Z\Sigma Z^* AZ,$$

see [27].

Thus, when Z is unitary, we have:

$$\nabla_Z F = 0 \iff AZ\Sigma - Z\Sigma Z^* AZ = 0 \iff AH - HA = 0 \iff$$

A and X have common eigenvectors, which leads us to the closed-form solution of the problem.

Theorem 6.3 *If matrix A is symmetric, then the solution to the following problem:*

$$\min_Z ||A - Z\Sigma Z^*||^2$$
$$ZZ^* = I$$

is given by $Z = Y$, where Y is the eigenvector matrix of A.

Note the minimum value of the function F is

$$F(Y) = ||Y(\Lambda - \Sigma)Y^*||^2 = ||\Lambda - \Sigma||^2 = \sum(\lambda_i - \sigma_i)^2.$$

The formula for the derivative (6.3.18) has been obtained disregarding the fact that $Z = col(X; X\Sigma)$; taking this fact into consideration, the formula of the derivative will be as follows:

$$F_Z = \begin{pmatrix} J_1 \\ J_1\Sigma \end{pmatrix}.$$

Having the analytical expression for the derivative, we can solve problem (6.3.16)-(6.3.17) using some gradient-based optimization algorithm, for example BFGS [60]. Once the solution matrix X has been found, the solution to the original problem (6.3.15) $\hat{C}$ and $\hat{K}$ can be found by substituting X into (6.3.12)-(6.3.13). We should note, that the computational cost of solving the minimization problem (6.3.16)-(6.3.17) is high. However, our numerical experiments show that a good approximation to the solution to the nearness problem (6.3.15) could be obtained by choosing X to be equal to the permuted eigenvector matrix of the pencil (M, C, K). The solution can be approximated by

$$\hat{C} \approx -MY_\sigma\Sigma^2U_2 \tag{6.3.19}$$

$$\hat{K} \approx -MY_\sigma\Sigma^2U_1, \tag{6.3.20}$$

where Y_σ is the permuted eigenvector matrix of the pencil (M, C, K) and the permutation σ is as defined in (6.1.5) and

$$U = \begin{pmatrix} Y_\sigma \\ Y_\sigma\Sigma \end{pmatrix}^{-1} = (U_1, U_2),\ U_1, U_2 \in \mathbb{C}^{2n\times n}.$$

6.4 Method of Alternating Projections

Now we turn our attention to the problem of choosing a good initial guess which can be used to compute a solution to Problem 1 using locally quadratically con-

vergent Newton's method. A good method to find a point close to a solution is **alternating projections method**.

To describe this method, consider two sets:

$$\mathcal{L} = \{(C, K) \in \mathbb{R}^{2n^2} \ | MX\Sigma^2 + CX\Sigma + KX = 0, \text{ for some matrix } X \text{ s.t } ||x_i|| = 1\},$$

and

$$\mathcal{A} = \{(C, K) \in \mathbb{R}^{2n^2} \mid C = C(\alpha) = C_0 + \sum_{i=1}^{n} \alpha_i C_i, \ K = K(\alpha) = K_0 + \sum_{i=1}^{n} \beta_i K_i\}.$$

The set $\mathcal{L}$ denotes the set of quadratic matrix pencils with given fixed matrix M as the leading matrix coefficient, and eigenvalues Σ. The set $\mathcal{A}$ denotes the set of quadratic pencils with matrix M as a leading coefficient and the matrices C, K are members of the corresponding affine families. In view of these definitions, Problem 1 can now be reformulated as follows.

Problem 2. Find $(\alpha, \beta) \in \mathbb{R}^{2n}$ such that $(M, C(\alpha), K(\beta)) \in \mathcal{L} \cap \mathcal{A}$.

Problem 2 can be solved using an alternating projections method [74]. The following results form a basis of such a method.

Theorem 6.4 *Let C_1, C_2 be closed convex sets in a finitely dimensional Hilbert space H, $C_1 \bigcap C_1 \neq \emptyset$ and let P_{C_1} and P_{C_2} denote projection operators onto C_1 and C_2, respectively. Then,*

$$\lim_{n\to\infty} (P_{C_1} P_{C_2})^n = \lim_{n\to\infty} (P_{C_2} P_{C_1})^n = P_{C_1 \cap C_2}.$$

Note, both of the sets $\mathcal{L}, \mathcal{A}$ are closed, but the set $\mathcal{L}$ is nonconvex. Thus, alternating projections might not converge. However, alternating projections never increases the distance between two successive iterates, as the following result shows.

Theorem 6.5 *Let C_1, C_2 be closed sets in a finitely dimensional Hilbert space H, $C_1 \bigcap C_1 \neq \emptyset$ and let $y \in C_2$. If*

$$x_1 = P_{C_1}(y), \ y_1 = P_{C_2}(x_1), \ x_2 = P_{C_1}(y_1),$$

then

$$||x_2 - y_1|| \leq ||x_1 - y_1|| \leq ||x_1 - y||.$$

Corollary 6.2 *For any given $x_0 \in H$, $\{(P_{C_1}P_{C_2})^n(x_0)\}_{n=0}^{\infty}$ is a nondecreasing sequence.*

As we noted above, the accumulation point of $\{(P_{C_1}P_{C_2})^n(x_0)\}_{n=0}^{\infty}$ is not necessarily a solution to Problem 2. However, in practice, even if alternating projections do not converge to the solution of Problem 2, an accumulation point is still close to a solution.

Now we need to derive projection operators $P_{\mathcal{L}}$ and $P_{\mathcal{A}}$. The inner product in R^{2n^2} will be defined in the standard way:

$$< (C_1, K_1), (C_2, K_2) >= trace(C_1^T C_2 + K_1^T K_2).$$

Let $P_{\mathcal{A}}(C, K) = (C(\hat{\alpha}), K(\hat{\beta}))$, then the coefficients $(\hat{\alpha}, \hat{\beta})$ are found as solutions of the following linear systems:

$$A_1\hat{\alpha} = b_1, \ A_2\hat{\beta} = b_2, \tag{6.4.21}$$

where $(A_1)_{ij} = trace(C_i^T C_j)$, $(A_2)_{ij} = trace(K_i^T K_j)$ and $(b_1)_i = trace((C - C_0)^T C_i)$, $(b_2)_i = trace((K - K_0)^T K_i)$. Define

$$c : \mathcal{A} \to R^{2n}, \ c(C, K) = (\alpha, \beta). \tag{6.4.22}$$

The projection onto $\mathcal{L}$ is the solution to the problem (6.3.11). Note, that the definition of the projection operator is different for the one usually used in numerical analysis and numerical linear algebra literature. The value of the operator could be found using a gradient search optimization routine or an approximation to the solution given in (6.3.19)-(6.3.20). Thus,

$$P_{\mathcal{L}}(C, K) \approx (MX_\sigma \Sigma^2 \overline{V}_2, MX_\sigma \Sigma^2 \overline{V}_1), \tag{6.4.23}$$

where X_σ is the permuted eigenvector matrix of the pencil (M, C, K) and the permutation σ is as defined in (6.1.5) and

$$\overline{V} = \begin{pmatrix} X_\sigma \\ X_\sigma \Sigma \end{pmatrix}^{-1} = (\overline{V}_1, \overline{V}_2),\ \overline{V}_1, \overline{V}_2 \in \mathbb{R}^{2n \times n}. \tag{6.4.24}$$

Our numerical experiments indicates that the convergence of alternating projections method does not suffer when the approximation of the projection operator defined above is used.

Algorithm 4 Alternating Projections Method

INPUT: λ^*, (α^0, β^0), ϵ
OUTPUT: $(\hat{\alpha}, \hat{\beta})$

1: **for** i = 0,1,... **do**
2: Form $(C(\alpha_i), K(\beta_i))$
3: Compute eigenvalues-eigenvectors of $(M, C(\alpha_i), K(\beta_i))$ and compute σ
4: Form matrix X_σ
5: Compute $(\hat{C}, \hat{K})$, approximation of projection of $(C(\alpha_i), K(\beta_i))$ onto $\mathcal{L}$, using (6.4.23)
6: Compute $(\alpha_{i+1}, \beta_{i+1})$, by solving (6.4.21)
7: Stop if $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon$
8: **end for**

As you can see from (6.4.23), (6.4.24) there is no need to compute the eigenvalues in order to compute the approximation of the projection operator. Only eigenvectors are needed. Instead of computing eigenvectors, we can approximate them. It can be done by using inverse iterations. Suppose that (α_i, β_i) is our current estimate of parameters, $X_\sigma^{(i)}$ is an approximation to $X_\sigma(\alpha_i, \beta_i)$, and the matrix of eigenvectors of $(M, C(\alpha_i), K(\beta_i))$, arranged in the "right" order (6.1.5). Let $x^i_{\sigma(j)}$ be the j^{th} column of $X_\sigma^{(i)}$. To find $(\alpha_{i+1}, \beta_{i+1})$, we compute $(\hat{C}, \hat{K})$:

$$(\hat{C}, \hat{K}) = (-MX_\sigma^{(i)}\Sigma^2 V_2, -MX_\sigma^{(i)}\Sigma^2 V_1), \tag{6.4.25}$$

where

$$V = \begin{pmatrix} X_\sigma^{(i)} \\ X_\sigma^{(i)} \Sigma \end{pmatrix}^{-1},$$

and then calculate $(\alpha_{i+1}, \beta_{i+1})$ by solving (6.4.21). To update our approximation to the eigenvectors, we apply one step of inverse iteration: we compute u_j, $j = 1, ..., 2n$:

$$\begin{pmatrix} 0 & I \\ K & C \end{pmatrix} \begin{pmatrix} u_j \\ z_j \end{pmatrix} = \mu_j \begin{pmatrix} I & 0 \\ 0 & -M \end{pmatrix} \begin{pmatrix} x^i_{\sigma(j)} \\ \mu_j x^i_{\sigma(j)} \end{pmatrix}, \ i = 1, ..., 2n. \tag{6.4.26}$$

We then define

$$x^{i+1}_{\sigma(j)} = \frac{u^j}{||u^j||},$$

which determines the new matrix X^{i+1}_{σ}. Vector u_j can be obtained as a solution of an $n \times n$ linear system (see [61]):

$$(\lambda^{*^2}_j M + \mu_j C + K)u_j = (C + 2\mu_j M)x^i_{\sigma(j)}. \tag{6.4.27}$$

Thus we are performing an alternating projections - like iteration where instead of computing the exact eigenvectors of $(M, C(\alpha), K(\beta))$ at each step and then computing permutation σ, we update an approximation to them by performing one step of inverse iterations.

Algorithm 5 Alternating Projections - like Method

INPUT: λ^*, (α^0, β^0), ϵ
OUTPUT: $(\hat{\alpha}, \hat{\beta})$

1: Form $(C(\alpha_0), K(\beta_0))$
2: Compute eigenvalues-eigenvectors of $(M, C(\alpha_0), K(\beta_0))$ and compute σ
3: Form matrix $X_\sigma^{(0)} = X_\sigma(\alpha_0, \beta_0)$
4: **for** i = 0,1,... **do**
5: Compute $(\hat{C}, \hat{K})$, quasi-projection of $(C(\alpha_i), K(\beta_i))$ onto $\mathcal{L}$, using (6.4.25)
6: Compute $(\alpha_{i+1}, \beta_{i+1})$, by solving (6.4.21)
7: Stop if $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon$
8: Form $(C, K) = (C(\alpha_{i+1}), K(\beta_{i+1}))$
9: Solve $2n$ linear systems

$$((\mu_j)^2 M + \mu_j C + K)u_j = (C + 2\mu_j M)x^i_{\sigma(j)},\ j = 1, ..., 2n$$

and compute

$$x^{i+1}_{\sigma(j)} = \frac{u^j}{||u^j||}$$

10: **end for**

6.5 Hybrid Method

On the basis of the discussion above, we state the following hybrid algorithm:

Algorithm 6 Hybrid Method

INPUT: λ^*, (α^0, β^0), ϵ_1, ϵ_2
OUTPUT: (α^*, β^*)

1: **while** $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon_1$ **do**
2: Form $J(\alpha^i, \beta^i)$ and compute $(\alpha^{i+1}, \beta^{i+1})$
3: **end while**
4: **while** $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon_2$ **do**
5: $(\alpha^{i+1}, \beta^{i+1}) = c((P_{\mathcal{A}} P_{\mathcal{L}})(C(\alpha^i), K(\beta^i)))$
6: **end while**

6.6 Numerical Experiments

Consider the mass-spring system 2.2 with 3 degrees for freedom; let $\alpha_0 = (1,1,1)$, $\beta_0 = (1,1,1)$ and

$$\lambda^* = \{-0.0271 \pm i1.0108, -0.0177 \pm i0.6724, -0.0023 \pm i0.2658\}.$$

After applying a few iterations of Algorithm 5, we obtain:

$$\alpha_{AP} = (0.0332, 0.0134, 0.0169),\ \beta_{AP} = (0.7188, 0.2193, 0.1915)$$

$$\lambda(\alpha_{AP}, \beta_{AP}) = \{-0.0271 \pm i1.0110, -0.0176 \pm i0.6724, -0.0021 \pm i0.2554\}.$$

Using $(\alpha_{AP},\ \beta_{AP})$ as initial guess for Newton's method, we obtain after several iterations the following improved values:

$$\alpha_N = (0.0139, 0.0203, 0.0199),\ \beta_N = (0.6038, 0.2722, 0.1988).$$

Verify:

$$||\lambda^* - \lambda(\alpha_N, \beta_N)|| = 1.29 \times 10^{-9}.$$

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