Reproducing Kernel Hilbert Space(RKHS)

Definition (Hilbert Space)

A Hilbert space is a complete inner product space.

Definition (Reproducing Kernel Hilbert Space)

Reproducing kernel Hilbert space is a Hilbert space where the evaluation functional is continuous.

Evaluation functional is defined as

$$\delta_x(f) = f(x)$$

A RKHS is generated by a Mercer kernel $K(x, y)$, which is symmetric and positive definite.

$$\int\int K(x, y)f(x)f(y)dxdy \geq 0, \quad \forall f$$

Reproducing Kernel Hilbert Space(RKHS)

Given a kernel K, let $K_x(\cdot)$ be the function obtained by fixing the first coordinate. That is, $K_x(y) = K(x, y)$. We can create functions by taking linear combinations of the kernel:

$$f(x) = \sum_{j=1}^{k} \alpha_j K_{x_j}(x)$$

Define following function space

$$\mathcal{H}_0 = \left\{ f : \sum_{j=1}^{k} \alpha_k K_{x_j}(x), k \geq 1, \alpha_j \in \mathbb{R}, x_j \in \mathbb{R} \right\}$$

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Given two such functions

$$f(x) = \sum_{j=1}^{k} \alpha_j K_{x_j}(x) \qquad g(x) = \sum_{j=1}^{m} \beta_j K_{y_j}(x)$$

Define an inner product

$$\langle f, g \rangle = \langle f, g \rangle_K = \sum_i \sum_j \alpha_i \beta_j K(x_i, y_j)$$

Which induce a norm

$$\|f\|_K = \sqrt{\langle f, f \rangle}$$

The completion of $\mathcal{H}_0$ with respect to $\| \cdot \|_K$ is denoted by $\mathcal{H}_K$ and is called the RKHS generated by K.

RKHS Regression and Classification

RKHS learning is under the regularization framework:

$$\min_{f \in \mathcal{H}_K} \frac{1}{n} \sum_{i=1}^{n} \ell(f(x_i), y_i) + \lambda \|f\|_K$$